Signali i sistemi

Signal $s(t)$ ima spektar $S(f)$ ograničen na opseg učestanosti $(0 \div f_m)$. Odabiranjem signala $s(t)$ dobijaju se 4 signala odbiraka:

$$
\begin{aligned}
s_1(t) &= s(t) \sum_{n=-\infty}^{\infty} \delta(t - kT_s), \\
s_2(t) &= s(t) \sum_{n=-\infty}^{\infty} \delta(t - kT_s - \tau_0), \\
s_3(t) &= s(t) \sum_{n=-\infty}^{\infty} \delta(t - kT_s - \tau_1) \text{ i} \\
s_4(t) &= s(t) \sum_{n=-\infty}^{\infty} \delta(t - kT_s - \tau_2),
\end{aligned}
$$

pri čemu je $T_s = \frac{2}{f_m}$ i $0 < \tau_0 < \tau_1 < \tau_2 < T_s/4$. Smatrati da je signal $s(t)$ realan. Da li je na osnovu spektara signala odbiraka ($s_i(t)$ $i = 1, 2, 3, 4$) moguće rekonstruisati spektar originalnog signala? Odgovor potkrepiti odgovarajućim dokazom.

Signals and Systems

The signal $s(t)$ is band-limited with the highest frequency component f_m. Sampling of the signal $s(t)$ produce 4 sampled signals:

$$\begin{aligned}
s_1(t) &= s(t)\sum_{n=-\infty}^{\infty}\delta(t-kT_s),\\
s_2(t) &= s(t)\sum_{n=-\infty}^{\infty}\delta(t-kT_s-\tau_0),\\
s_3(t) &= s(t)\sum_{n=-\infty}^{\infty}\delta(t-kT_s-\tau_1) \text{ and}\\
s_4(t) &= s(t)\sum_{n=-\infty}^{\infty}\delta(t-kT_s-\tau_2),
\end{aligned}$$

such that: $T_s = \frac{2}{f_m}$ and $0 < \tau_0 < \tau_1 < \tau_2 < T_s/4$. Assume that the signal $s(t)$ is real.

Is it possible to reconstruct spectrum of the signal $s(t)$ using the sampled signals ($s_i(t)$ $i = 1, 2, 3, 4$)? The answer should contain an explanation.

Rešenje

Spektar signala $s_2(t)$ je:

$$\begin{aligned}
S_2(f) &= \int_{-\infty}^{\infty} s(t) \sum_{n=-\infty}^{\infty} \delta(t - kT_s - \tau_0) e^{-j2\pi f t} dt \\
&= \int_{-\infty}^{\infty} s(t) \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{jk\frac{2\pi}{T_s}(t-\tau_0)} e^{-j2\pi f t} dt \\
&= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{-jk\frac{2\pi}{T_s}\tau_0} \int_{-\infty}^{\infty} s(t) e^{-j2\pi(f-\frac{k}{T_s})t} dt \\
&= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{-jk\frac{2\pi}{T_s}\tau_0} S(f - \frac{k}{T_s})
\end{aligned}$$

Pri traženju spektra $S_2(f)$ iskorišćena je jednakost:

$$\sum_{k=-\infty}^{\infty} \delta(t - kT_s - \tau_0) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{jk\frac{2\pi}{T_s}(t-\tau_0)}$$

do koje se može doći razvojem povorke delta impulsa u FR. Spektri ostalih signala se dobija na osnovu spektra signala $s_2(t)$ jednostavnom zamenom $\tau_0 = 0$, $\tau_0 = \tau_1$ i $\tau_0 = \tau_2$ te se dobijaju sledeće jednačine:

$$S_1(f) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} S(f - \frac{k}{T_s})$$

$$S_3(f) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{-jk\frac{2\pi}{T_s}\tau_1} S(f - \frac{k}{T_s})$$

$$S_4(f) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{-jk\frac{2\pi}{T_s}\tau_2} S(f - \frac{k}{T_s})$$

Zbog hermitske simetrije spektra signala $s(t)$ i periodičnosti spektara signala odbiraka dovoljno je posmatrati opseg učestanosti $(0 \div f_m)$.

Na opsegu od $(0 \div f_m/2)$ dobijaju se sledeći spektri:

$$\begin{aligned}
S_1(f) &= \frac{f_m}{2} (S(f + f_m/2) + S(f) + S(f - f_m/2) + S(f - f_m)) \\
S_2(f) &= \frac{f_m}{2} \left(S(f + f_m/2) e^{j\frac{2\pi}{T_s}\tau_0} + S(f) + S(f - f_m/2) e^{-j\frac{2\pi}{T_s}\tau_0} + S(f - f_m) e^{-j\frac{4\pi}{T_s}\tau_0} \right) \\
S_3(f) &= \frac{f_m}{2} \left(S(f + f_m/2) e^{j\frac{2\pi}{T_s}\tau_1} + S(f) + S(f - f_m/2) e^{-j\frac{2\pi}{T_s}\tau_1} + S(f - f_m) e^{-j\frac{4\pi}{T_s}\tau_1} \right) \\
S_4(f) &= \frac{f_m}{2} \left(S(f + f_m/2) e^{j\frac{2\pi}{T_s}\tau_2} + S(f) + S(f - f_m/2) e^{-j\frac{2\pi}{T_s}\tau_2} + S(f - f_m) e^{-j\frac{4\pi}{T_s}\tau_2} \right)
\end{aligned}$$

odakle sledi:

$$S(f) = \frac{2}{f_m} \frac{B_1 S_1(f) + B_2 S_2(f) + B_3 S_3(f) + B_4 S_4(f)}{A}$$

gde je:

$$\begin{aligned}
B_1 &= \Big(e^{j\frac{2\pi(-\tau_0+\tau_1-2\tau_2)}{T_s}} - e^{j\frac{2\pi(-\tau_0-2\tau_1+\tau_2)}{T_s}} - e^{j\frac{2\pi(\tau_0-\tau_1-2\tau_2)}{T_s}} + e^{j\frac{2\pi(\tau_0-2\tau_1-\tau_2)}{T_s}} + \\
&\quad + e^{j\frac{2\pi(-2\tau_0-\tau_1+\tau_2)}{T_s}} + e^{j\frac{2\pi(-2\tau_0+\tau_1-\tau_2)}{T_s}} \Big) \\
B_2 &= j\left(\sin\left(\frac{2\pi(\tau_1-\tau_2)}{T_s}\right) + 2e^{j\frac{4\pi\tau_1}{T_s}} \sin\left(\frac{2\pi\tau_2}{T_s}\right) - 2e^{j\frac{4\pi\tau_2}{T_s}} \sin\left(\frac{2\pi\tau_1}{T_s}\right)\right) \\
B_3 &= j\left(\sin\left(\frac{2\pi(\tau_2-\tau_0)}{T_s}\right) - 2e^{j\frac{4\pi\tau_0}{T_s}} \sin\left(\frac{2\pi\tau_2}{T_s}\right) + 2e^{-j\frac{4\pi\tau_2}{T_s}} \sin\left(\frac{2\pi\tau_0}{T_s}\right)\right) \\
B_4 &= j\left(\sin\left(\frac{2\pi(\tau_0-\tau_1)}{T_s}\right) + 2e^{-j\frac{4\pi\tau_0}{T_s}} \sin\left(\frac{2\pi\tau_1}{T_s}\right) - 2e^{-j\frac{4\pi\tau_1}{T_s}} \sin\left(\frac{2\pi\tau_0}{T_s}\right)\right)
\end{aligned}$$

$$\begin{aligned}
A &= j\left(\sin\left(\frac{2\pi(\tau_0-\tau_1)}{T_s}\right) + \sin\left(\frac{2\pi(\tau_2-\tau_0)}{T_s}\right) + \sin\left(\frac{2\pi(\tau_1-\tau_2)}{T_s}\right)\right) \\
&\quad + e^{j\frac{2\pi(-\tau_0+\tau_1-2\tau_2)}{T_s}} - e^{j\frac{2\pi(-\tau_0-2\tau_1+\tau_2)}{T_s}} - e^{j\frac{2\pi(\tau_0-\tau_1-2\tau_2)}{T_s}} + e^{j\frac{2\pi(\tau_0-2\tau_1-\tau_2)}{T_s}} + \\
&\quad + e^{j\frac{2\pi(-2\tau_0-\tau_1+\tau_2)}{T_s}} - e^{j\frac{2\pi(-2\tau_0+\tau_1-\tau_2)}{T_s}} + \\
&\quad + 2j\left(e^{-j\frac{4\pi\tau_0}{T_s}}\left(\sin\left(\frac{2\pi\tau_1}{T_s}\right) - \sin\left(\frac{2\pi\tau_2}{T_s}\right)\right) + e^{-j\frac{4\pi\tau_1}{T_s}}\left(\sin\left(\frac{2\pi\tau_2}{T_s}\right) - \sin\left(\frac{2\pi\tau_0}{T_s}\right)\right)\right. \\
&\quad \left. + e^{-j\frac{4\pi\tau_2}{T_s}}\left(\sin\left(\frac{2\pi\tau_0}{T_s}\right) - \sin\left(\frac{2\pi\tau_2}{T_s}\right)\right)\right)
\end{aligned}$$

Na opsegu od $(f_m/2 \div f_m)$ dobijaju se sledeći spektri:

$$\begin{aligned}
S_1(f) &= \frac{f_m}{2}\left(S(f) + S(f-f_m/2) + S(f-f_m) + S(f-3f_m/2)\right) \\
S_2(f) &= \frac{f_m}{2}\left(S(f) + S(f-f_m/2)e^{-j\frac{2\pi}{T_s}\tau_0} + S(f-f_m)e^{-j\frac{4\pi}{T_s}\tau_0} + S(f-3f_m/2)e^{-j\frac{6\pi}{T_s}\tau_0}\right) \\
S_3(f) &= \frac{f_m}{2}\left(S(f) + S(f-f_m/2)e^{-j\frac{2\pi}{T_s}\tau_1} + S(f-f_m)e^{-j\frac{4\pi}{T_s}\tau_1} + S(f-3f_m/2)e^{-j\frac{6\pi}{T_s}\tau_1}\right) \\
S_4(f) &= \frac{f_m}{2}\left(S(f) + S(f-f_m/2)e^{-j\frac{2\pi}{T_s}\tau_2} + S(f-f_m)e^{-j\frac{4\pi}{T_s}\tau_2} + S(f-3f_m/2)e^{-j\frac{6\pi}{T_s}\tau_2}\right)
\end{aligned}$$

odakle sledi:

$$S(f) = \frac{2}{f_m} \frac{B_5 S_1(f) + B_6 S_2(f) + B_7 S_3(f) + B_8 S_4(f)}{A_1}$$

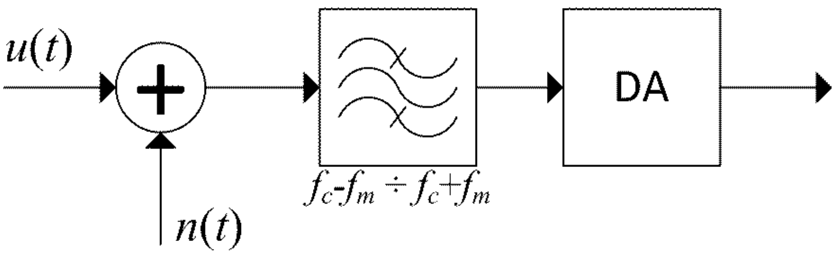
gde je:

$$\begin{aligned}
B_5 &= e^{j\frac{2\pi\tau_0}{T_s}}\left(e^{j\frac{4\pi\tau_1}{T_s}}-e^{j\frac{4\pi\tau_2}{T_s}}\right)-e^{j\frac{2\pi\tau_1}{T_s}}\left(e^{j\frac{4\pi\tau_0}{T_s}}-e^{j\frac{4\pi\tau_2}{T_s}}\right)+e^{j\frac{2\pi\tau_2}{T_s}}\left(e^{j\frac{4\pi\tau_0}{T_s}}-e^{j\frac{4\pi\tau_1}{T_s}}\right)\\
B_6 &= e^{j\frac{6\pi\tau_0}{T_s}}\left(e^{j\frac{2\pi\tau_1}{T_s}}-e^{j\frac{2\pi\tau_2}{T_s}}\right)\left(e^{j\frac{2\pi\tau_1}{T_s}}-1\right)\left(e^{j\frac{2\pi\tau_2}{T_s}}-1\right)\\
B_7 &= -e^{j\frac{6\pi\tau_1}{T_s}}\left(e^{j\frac{2\pi\tau_0}{T_s}}-e^{j\frac{2\pi\tau_2}{T_s}}\right)\left(e^{j\frac{2\pi\tau_0}{T_s}}-1\right)\left(e^{j\frac{2\pi\tau_2}{T_s}}-1\right)\\
B_8 &= e^{j\frac{6\pi\tau_2}{T_s}}\left(e^{j\frac{2\pi\tau_0}{T_s}}-e^{j\frac{2\pi\tau_1}{T_s}}\right)\left(e^{j\frac{2\pi\tau_0}{T_s}}-1\right)\left(e^{j\frac{2\pi\tau_1}{T_s}}-1\right)\\
A_1 &= e^{-j\frac{6\pi(\tau_0+\tau_1+\tau_2)}{T_s}}\left(e^{j\frac{2\pi\tau_0}{T_s}}-e^{j\frac{2\pi\tau_1}{T_s}}\right)\left(e^{j\frac{2\pi\tau_0}{T_s}}-e^{j\frac{2\pi\tau_2}{T_s}}\right)\left(e^{j\frac{2\pi\tau_1}{T_s}}-e^{j\frac{2\pi\tau_2}{T_s}}\right)\\
&\quad \left(e^{j\frac{2\pi\tau_0}{T_s}}-1\right)\left(e^{j\frac{2\pi\tau_1}{T_s}}-1\right)\left(e^{j\frac{2\pi\tau_2}{T_s}}-1\right)
\end{aligned}$$

Analogne modulacije

Na slici 1 prikazana je blok šema prijemnika KAM signala sa idealnim detektorom anvelope. Na ulaz prijemnika dolazi KAM signal kome je pridodat šum. Indeks modulacije je m_0, amplituda nosioca je U_c, a učestanost nosioca je f_c. Modulišući signal $m(t)$ ima spektar u opsegu $(0 \div f_m)$ i srednju snagu P_m. Spektralna gustina srednje snage šuma data je izrazom $S_n(f) = 1/(f^2 + a^2)$ (gde je a pozitivna realna konstanta). Odrediti analitički izraz za odnos signal šum na izlazu prijemnika. Prilikom izvođenja smatrati da je snaga korisnog signala mnogo veća od snage šuma, da sinusna i kosinusna komponenta uskopojasnog šuma imaju istu srednju snagu i da su međusobno nekorelisane.

Pomoć: Za x blisko nuli važi $\sqrt{1+x} \approx 1 + \frac{x}{2}$.



Slika 1: Prijemnik sa detektorom anvelope

Analog Modulations

A block scheme of AM signal receiver with an ideal envelope detector is shown in Fig. 1. The input signal is AM signal with additive noise. The modulation index is m_0, the carrier amplitude is U_c, and the carrier frequency is f_c. The modulating signal $m(t)$ has limited spectrum up to f_m and the power P_m. The power spectrum density of noise signal is $S_n(f) = 1/(f^2 + a^2)$ (where a is positive real constant). Find an analytic expression for signal to noise ratio on the receiver output. Assume the following: the power of noise is small compared to the power of signal, sine and cosine components of bandpass noise have the same power and they are mutually uncorrelated.

Hint: If x is close to 0 than $\sqrt{1+x} \approx 1 + \frac{x}{2}$.

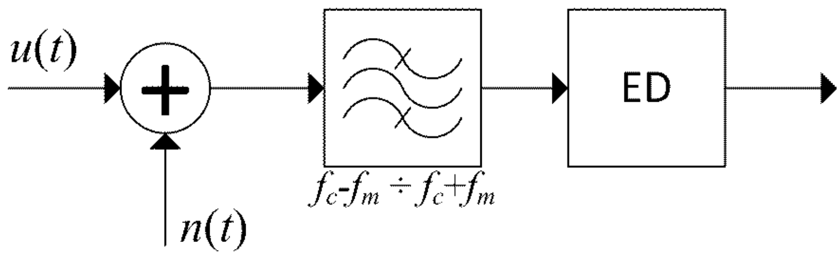


Figure 1: A receiver with envelope detector

Rešenje

Signal na ulazu prijemnika je:

$$u(t) = U_c\left(1 + m_0 m(t)\right)\cos(2\pi f_c t) + n(t)$$

Kroz pojasni filtar prolazi celokupni modulisani signal, dok šum postaje uskopojasni, čija spektralna gustina srednje snage postaje:

$$S_{n,c}(f) = \begin{cases} 1/(f^2 + a^2) & \text{za} f_c - f_m < |f| \leq f_c + f_m \\ 0 & \text{inače} \end{cases}$$

Signal na izlazu pojasnog filtra se može prikazati kao:

$$u_1(t) = U_c\left(1 + m_0 m(t)\right)\cos(2\pi f_c t) + n_c(t)\cos(2\pi f_c t) + n_s(t)\sin(2\pi f_c t) = A(t)\cos(2\pi f_c t + \Phi(t))$$

Anvelopa ovog signala ima oblik:

$$\begin{aligned} A(t) &= \sqrt{\left(U_c\left(1 + m_0 m(t)\right) + n_c(t)\right)^2 + n_s^2(t)} \\ &= U_c\left(1 + m_0 m(t)\right)\sqrt{1 + \frac{2n_c(t)}{U_c\left(1 + m_0 m(t)\right)} + \frac{n_c^2(t)}{U_c^2\left(1 + m_0 m(t)\right)^2} + \frac{n_s^2(t)}{U_c^2\left(1 + m_0 m(t)\right)^2}} \\ &\approx U_c\left(1 + m_0 m(t)\right)\sqrt{1 + \frac{2n_c(t)}{U_c\left(1 + m_0 m(t)\right)}} \\ &\approx U_c\left(1 + m_0 m(t)\right) + n_c(t) \end{aligned}$$

Poslednja dva člana pod korenom su zanemarena pošto je snaga signala mnogo veća od snage šuma. To je iskorišćeno i pri drugoj aproksimaciji gde je koren zamenjen sa prva dva člana Tejlorovog reda.

Potrebno je odrediti snagu kosinusne komponente uskopojasnog šuma.

Autokorelacija šuma je:

$$R_N(\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} n(t)n(t+\tau)dt = \overline{n(t)n(t+\tau)}$$

Uvrštavanjem $n(t) = n_c(t)\cos(2\pi f_c t) + n_s(t)\sin(2\pi f_c t)$ u izraz za autokorelaciju dobija se:

$$\begin{aligned} R_N(\tau) &= \overline{n_c(t)n_c(t+\tau)}\cdot\overline{\cos\left(2\pi f_c t\right)\cos\left(2\pi f_c(t+\tau)\right)} \\ &\quad+\overline{n_c(t)n_s(t+\tau)}\cdot\overline{\cos\left(2\pi f_c t\right)\sin\left(2\pi f_c(t+\tau)\right)} \\ &\quad+\overline{n_s(t)n_c(t+\tau)}\cdot\overline{\sin\left(2\pi f_c t\right)\cos\left(2\pi f_c(t+\tau)\right)} \\ &\quad+\overline{n_s(t)n_s(t+\tau)}\cdot\overline{\sin\left(2\pi f_c t\right)\sin\left(2\pi f_c(t+\tau)\right)} \\ &= \overline{n_c(t)n_c(t+\tau)}\cdot\frac{1}{2}\cos\left(2\pi f_c\tau\right) + \overline{n_s(t)n_s(t+\tau)}\cdot\frac{1}{2}\cos\left(2\pi f_c\tau\right) \end{aligned}$$

Pri izvođenju je iskoršćen uslov da su kosinusna i sinusna komponenta uskopojasnog šuma nekorelisane $\overline{n_c(t)n_s(t+\tau)} = \overline{n_s(t)n_c(t+\tau)} = 0$, kao i da je $\overline{\cos(2\pi f_c t)\cos(2\pi f_c(t+\tau))} = \overline{\sin(2\pi f_c t)\sin(2\pi f_c(t+\tau))} = \frac{1}{2}\cos(2\pi f_c \tau)$

Po uslovu zadatka važi da je $\overline{n_c(t)^2} = \overline{n_s(t)^2}$ i pošto važi

$$R_N(0) = \frac{1}{2}\overline{n_c(t)^2} + \frac{1}{2}\overline{n_s(t)^2}$$

sledi:

$$R_N(0) = \overline{n_c(t)^2} = \overline{n_s(t)^2}$$

Tako da se od izraza za srednju snagu šuma dolazi na osnovu izraza za srednju snagu uskpojasnog šuma koja je u ovom slučaju:

$$\overline{n_c(t)^2} = R_N(0) = 2\int_{f_c-f_m}^{f_c+f_m} \frac{1}{f^2+a^2} df = \frac{2}{a}\left(\arctan\frac{f_c+f_m}{a} - \arctan\frac{f_c-f_m}{a}\right)$$

Odnosno korišćenjem odgovarajuće trigonometrijske jednačine

$$\overline{n_c(t)^2} = \frac{2}{a}\arctan\frac{2af_m}{a^2+f_c^2-f_m^2}$$

Tražen odnosn signal šum je:

$$\mathrm{SNR} = \frac{P_s}{\overline{n_c(t)^2}} = \frac{U_c^2 m_0^2 P_m}{\frac{2}{a}\arctan\frac{2af_m}{a^2+f_c^2-f_m^2}}$$

Digitalne telekomunikacije / Digital communications

Zadatak:

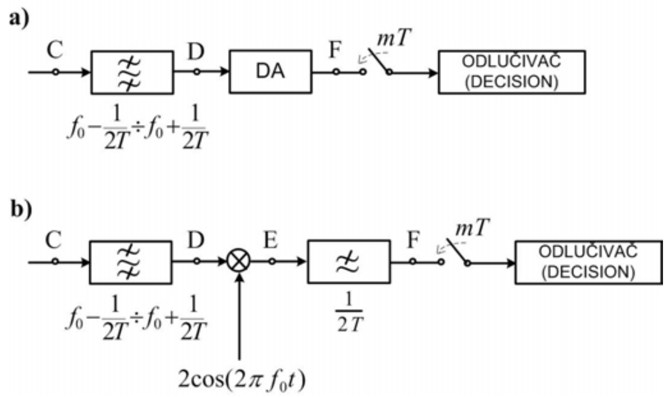
Posmatra se prenos ASK signala oblika $s_m(t) = s(t)\cos(2\pi f_0 t)$, gde je

$$s(t) = 2TU \sum_k a_k h(t - kT)$$

binarni modulišući signal sa periodičnom informacionom sekvencom $\{\ldots, 1, 0, 1, 0, 1, \ldots\}$ i elementarnim impulsom $h(t)$ čiji je spektar oblika:

$$H(f) = \begin{cases} \cos\left(\frac{\pi f T}{2}\right), & |f| \le \frac{1}{T} \\ 0, & \text{inače} \ . \end{cases}$$

Učestanost nosioca je $f_0 = 2/T$. Odrediti i skicirati signal (u vremenskom domenu) u tački F na izlazu prijemnika u slučaju nekoherentnog (a), odnosno koherentnog (b) prijema. Odrediti marginu šuma u oba slučaja ako je prag odlučivanja u odlučivaču U. Objasniti koji od ova dva prijemnika obezbeđuje manju verovatnoću greške.



Problem:

Consider an ASK signal of the form $s_m(t) = s(t)\cos(2\pi f_0 t)$, where

$$s(t) = 2TU \sum_k a_k h(t - kT)$$

is a binary modulating signal with periodic information sequence $\{\ldots, 1, 0, 1, 0, 1, \ldots\}$ and a pulse $h(t)$ whose spectrum is given by:

$$H(f) = \begin{cases} \cos\left(\frac{\pi f T}{2}\right), & |f| \le \frac{1}{T} \\ 0, & \text{elsewhere} \ . \end{cases}$$

Carrier frequency is $f_0 = 2/T$. Determine and depict the signal (in the time domain) at the point F at the receiver's output in case of noncoherent (a) and coherent (b) detection. Determine the noise margin (the largest absolute value of the amplitude of the noise that does not cause an error) in both cases if the decision threshold is set to U. Explain which of the two receivers ensures smaller error probability.

Rešenje:

Uzimajući u obzir oblik informacione sekvence, modulišući signal se može predstaviti u obliku

$$s(t) = 2TU \sum_k h(t - 2kT),$$

odakle se izračunava njegov spektar kao

$$S(f) = 2TUH(f) \frac{1}{2T} \sum_k \delta\left(f - \frac{k}{2T}\right) = UH(f) \sum_k \delta\left(f - \frac{k}{2T}\right).$$

Vidi se da je

$$s(t) = U + U\sqrt{2}\cos\left(\frac{\pi t}{T}\right),$$

tj.

$$s_m(t) = \left(U + U\sqrt{2}\cos\left(\frac{\pi t}{T}\right)\right)\cos(2\pi f_0 t).$$

U slučaju nekoherentnog prijema (a) signal u tački F biće:

$$s_F(t) = U\left|1 + \sqrt{2}\cos\left(\frac{\pi t}{T}\right)\right|$$

pa je njegova margina šuma $U(2 - \sqrt{2}) \approx 0.59U$.

U slučaju konerentne detekcije imamo

$$s_F(t) = U\left(1 + \sqrt{2}\cos\left(\frac{\pi t}{T}\right)\right),$$

a margina šuma je u tom slučaju $U\sqrt{2} \approx 1.41U$.

Zbog veće margine šuma, manju verovatnoću greške obezbeđuje koherentna detekcija.

Statistička teorija telekomunikacija / Statistical theory of communications

Zadatak:

Kroz kanal sa slabljenjem $\mathbf{U}$ i kašnjenjem $\mathbf{V}$ poslat je impuls

$$p(t) = \begin{cases} 1\,, & 0 \leq t < T \\ 0\,, & \text{inače} \end{cases},$$

tako da je signal na prijemu oblika

$$\mathbf{X}(t) = \mathbf{U}p(t - \mathbf{V}).$$

Ako su $\mathbf{U}$ i $\mathbf{V}$ nezavisne slučajne promenljive sa uniformnom, odnosno eksponencijalnom raspodelom:

$$\varphi_{\mathbf{U}}(u) = \begin{cases} 1/\Delta, & 0 \leq u \leq \Delta \\ 0, & \text{inače}, \end{cases} \qquad \varphi_{\mathbf{V}}(v) = \begin{cases} \lambda e^{-\lambda v}, & v \geq 0 \\ 0, & \text{inače}, \end{cases}$$

odrediti kumulativnu funkciju raspodele verovatnoće amplitude primljenog signala $\mathbf{X}(t)$. Ako prijemnik može da detektuje samo signal amplitude veće od δ, pri čemu je $0 < \delta < \Delta$, izračunati verovatnoću detekcije poslatog impulsa u trenutku t: $\mathbb{P}[\mathbf{X}(t) > \delta]$. Odrediti optimalan trenutak detekcije t^* u kome je ova verovatnoća maksimizovana.

Problem:

Electrical signal

$$p(t) = \begin{cases} 1\,, & 0 \leq t < T \\ 0\,, & \text{elsewhere} \end{cases}$$

is sent through the channel with attenuation $\mathbf{U}$ and delay $\mathbf{V}$, so that the received signal is of the form

$$\mathbf{X}(t) = \mathbf{U}p(t - \mathbf{V}).$$

If $\mathbf{U}$ and $\mathbf{V}$ are independent random variables with uniform and exponential distribution, respectively:

$$\varphi_{\mathbf{U}}(u) = \begin{cases} 1/\Delta, & 0 \leq u \leq \Delta \\ 0, & \text{elsewhere} \end{cases}, \qquad \varphi_{\mathbf{V}}(v) = \begin{cases} \lambda e^{-\lambda v}, & v \geq 0 \\ 0, & \text{elsewhere} \end{cases},$$

find the cumulative distribution function of the received signal's amplitude $\mathbf{X}(t)$. If the receiver is able to detect only voltages greater than δ, where $0 < \delta < \Delta$, calculate the probability of detection of the sent impulse at time t: $\mathbb{P}[\mathbf{X}(t) > \delta]$. Determine the optimal moment of detection t^* at which this probability is maximized.

Rešenje:

Potrebno je odrediti $F_{\mathbf{X}(t)}(x) = \mathbb{P}[\mathbf{X}(t) \leq x]$. Kako je očigledno $0 \leq \mathbf{X}(t) \leq \Delta$ za svako t, dobija se $F_{\mathbf{X}(t)}(x) = 0$ za $x < 0$ i $F_{\mathbf{X}(t)}(t) = 1$ za $x \geq \Delta$. Neka je nadalje $0 \leq x < \Delta$.

U tom slučaju događaj $\mathbf{X}(t) \leq x$ se može izraziti na sledeći način

$$\begin{aligned}\big\{\mathbf{X}(t) \leq x\big\} &= \big\{p(t-\mathbf{V}) = 0\big\} \cup \big\{p(t-\mathbf{V}) = 1,\ \mathbf{U} \leq x\big\} \\ &= \big\{t < \mathbf{V}\big\} \cup \big\{t > \mathbf{V} + T\big\} \cup \big\{\mathbf{V} < t < \mathbf{V} + T,\ \mathbf{U} \leq x\big\}\end{aligned}$$

pa je

$$F_{\mathbf{X}(t)}(x) = \mathbb{P}[\mathbf{V} > t] + \mathbb{P}[\mathbf{V} < t - T] + \mathbb{P}[t - T < \mathbf{V} < t] \cdot \mathbb{P}[\mathbf{U} \leq x].$$

Treba razmotriti dva slučaja, $t < T$ i $t \geq T$. U prvom slučaju se dobija

$$F_{\mathbf{X}(t)}(x) = e^{-\lambda t} + \big(1 - e^{-\lambda t}\big)\frac{x}{\Delta},$$

a u drugom

$$F_{\mathbf{X}(t)}(x) = e^{-\lambda t} + \big(1 - e^{-\lambda(t-T)}\big) + \big(e^{-\lambda(t-T)} - e^{-\lambda t}\big)\frac{x}{\Delta},$$

tako da je konačno

$$F_{\mathbf{X}(t)}(x) = \begin{cases} 0, & x < 0 \\ e^{-\lambda t} + \big(1 - e^{-\lambda t}\big)\Delta^{-1}x, & 0 \leq x < \Delta, \quad t < T \\ e^{-\lambda t} + \big(1 - e^{-\lambda(t-T)}\big) + \big(e^{-\lambda(t-T)} - e^{-\lambda t}\big)\Delta^{-1}x, & 0 \leq x < \Delta, \quad t \geq T \\ 1, & x \geq \Delta \quad . \end{cases}$$

Verovatnoća detekcije impulsa u trenutku t sada se lako određuje kao $\mathbb{P}[\mathbf{X}(t) > \delta] = 1 - F_{\mathbf{X}(t)}(\delta)$, tj.

$$\mathbb{P}[\mathbf{X}(t) > \delta] = \begin{cases} \big(1 - e^{-\lambda t}\big)\big(1 - \frac{\delta}{\Delta}\big), & t < T \\ e^{-\lambda t}\big(e^{\lambda T} - 1\big)\big(1 - \frac{\delta}{\Delta}\big), & t \geq T. \end{cases}$$

Vidi se da ova verovatnoća, posmatrana kao funkcija vremena, raste u intervalu $0 \leq t \leq T$, nakon čega monotono opada. Optimalan trenutak detekcije u kome je ona maksimizovana je stoga $t^* = T$.

Teorija informacija / Information theory

Zadatak:

3 crne i 3 bele kuglice, koje se razlikuju jedino po boji, nalaze se u 2 posude, tako da su u svakoj posudi po 3 kuglice (ne obavezno iste boje). U jednom potezu se iz prve posude na slučaj izvlači jedna kuglica i prebacuje u drugu, a posle toga se iz druge posude na slučaj izvlači jedna kuglica i prebacuje u prvu. Ako se ovaj postupak ponavlja mnogo puta, odrediti prosečnu količinu informacije koju sadrži podatak o tome koje kuglice se nalaze u prvoj posudi nakon jednog poteza.

Problem:

3 black and 3 white balls, which differ only in colour, are put in 2 bowls, so that there are 3 balls in each bowl (not necessarily of the same colour). In one move, one ball from the first bowl is chosen at random and put into the second bowl, and then one ball chosen at random from the second ball is put into the first bowl. If this is repeated many times, calculate the average amount of information conveyed by the contents of the first bowl after a single move.

Rešenje:

Neka su b' i b'' brojevi belih kuglica u prvoj posudi pre i posle jednog poteza. Ako su sa C i B označeni događaji da je izvučena crna ili bela kuglica, sa I i II prvo i drugo izvlačenje u jednom potezu, a sa p_{I} i p_{II} njihove verovatnoće, onda su mogući sledeći potezi sa odgovarajućim verovatnoćama:

b'	I	II	p_{I}	p_{II}	p	b''
0	C	C	1	$\frac{1}{4}$	$\frac{1}{4}$	0
0	C	B	1	$\frac{3}{4}$	$\frac{3}{4}$	1
1	B	C	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{12}$	0
1	B	B	$\frac{1}{3}$	$\frac{3}{4}$	$\frac{1}{4}$	1
1	C	C	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	1
1	C	B	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	2
2	B	C	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	1
2	B	B	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	2
2	C	C	$\frac{1}{3}$	$\frac{3}{4}$	$\frac{1}{4}$	2
2	C	B	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{12}$	3
3	B	C	1	$\frac{3}{4}$	$\frac{3}{4}$	2
3	B	B	1	$\frac{1}{4}$	$\frac{1}{4}$	3

Pri određivanju verovatnoća, uzeto je u obzir koliko kojih kuglica ima u posudi iz koje se neka od njih izvlači, kao i da su izvlačenja u jednom potezu nezavisna (osim preko brojeva kuglica).

Proces dobijen uzastopnim izvlačenjima je Markovljev izvor, čije stanje je npr. broj belih kuglica u prvoj posudi. Na osnovu prethodne tabele, uzimajući u obzir da su različiti potezi za isto polazno stanje disjunktni događaji, sabiranjem verovatnoća onih koji dovode u isto odredišno stanje dobija se matrica verovatnoća prelaza izvora,

$$\Pi = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ \frac{1}{12} & \frac{7}{12} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{7}{12} & \frac{1}{12} \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}.$$

Rešenje problema je entropija ovog izvora. Da bi se ona izračunala, potrebno je odrediti vektor njegovih stacionarnih verovatnoća

$$\mathbf{t} = \begin{bmatrix} t_0 & t_1 & t_2 & t_3 \end{bmatrix},$$

koji je rešenje sistema jednačina

$$\mathbf{t}\Pi = \mathbf{t},$$
$$t_0 + t_1 + t_2 + t_3 = 1.$$

Iz simetrije izvora može da se zaključi da je $t_0 = t_3$ i $t_1 = t_2$, na osnovu čega se lako dobija

$$t_0 = t_3 = \frac{1}{20},$$
$$t_1 = t_2 = \frac{9}{20}.$$

Entropije prelaza iz pojedinih stanja su

$$H_0 = H_3 = \frac{1}{4}\,\mathrm{ld}\,4 + \frac{3}{4}\,\mathrm{ld}\,\frac{4}{3} = 2 - \frac{3}{4}\,\mathrm{ld}\,3$$

i

$$H_1 = H_2 = \frac{1}{12}\,\mathrm{ld}\,12 + \frac{7}{12}\,\mathrm{ld}\,\frac{12}{7} + \frac{1}{3}\,\mathrm{ld}\,3 = \frac{4}{3} + \mathrm{ld}\,3 - \frac{7}{12}\,\mathrm{ld}\,7,$$

odakle sledi da je entropija izvora

$$H = t_0 H_0 + t_1 H_1 + t_2 H_2 + t_3 H_3 = \frac{7}{5} + \frac{33}{40}\,\mathrm{ld}\,3 - \frac{21}{40}\,\mathrm{ld}\,7 \approx 1.2337.$$