1 Analiza signala i sistema
Signals and systems

Zadatak:

OFDM sistem (*Orthogonal Frequency Division Multiplexing*) prenosi N digitalnih signala $s_i(t), i = 1, 2, \ldots, N$, upotrebom N frekvencijskih nosilaca međusobno pomerenih u frekvencijskom domenu za učestanost $\triangle f$. Kompleksna predstava signala $s_i(t)$ i-tog frekvencijskog kanala u osnovnom opsegu je:

$$s_i(t) = x_i \cdot e^{2\pi j \cdot \triangle f \cdot i \cdot t}, \quad 0 \leq t \leq T, \tag{1}$$

gde su x_i kompleksni simboli modulacionog alfabeta koji se koristi (npr. m-QAM, m-PSK, itd.), a T je period odnosno trajanje jednog OFDM simbola. Vremenski oblik jednog OFDM simbola trajanja T dobija se kao suma svih N komponentnih signala:

$$s(t) = \sum_{i=1}^{N} s_i(t) = \sum_{i=1}^{N} x_i \cdot e^{2\pi j \cdot \triangle f \cdot i \cdot t}, \quad 0 \leq t \leq T. \tag{2}$$

a) Odabrati minimalno frekvencijsko rastojanje $\triangle f$ susednih komponentnih signala $s_i(t)$ takvo da su bilo koja dve različita komponentna signala $s_i(t)$ i $s_k(t)$, $i \neq k$, ortogonalna u vremenskom domenu. (25)

b) Odabrati skup N učestanosti f_i na kojima će biti formiran OFDM signal $s(t)$ u osnovnom opsegu tako da širina spektra OFDM signala bude minimalna. (10)

c) Na ulaz OFDM predajnika dolazi niz informacionih simbola $\{x_k\}$, $k = 1, 2, \ldots$, modulacionog alfabeta čije je trajanje simbolskog intervala $T_S = \frac{T}{N}$. Tokom trajanja jednog OFDM simbolskog intervala T prihvata se N modulacionih simbola na ulazu i formnira jedan OFDM simbol na izlazu OFDM predajnika. Pokazati da se vremenski odbirci OFDM signala $s(t)$ u trenucima $t = k \cdot T_S$ mogu predstaviti kao inverzna diskretna Furijeova transformacija (IDFT) povorke ulaznih modulacionih simbola $\{x_k\}$. Da li su odbirci $\{s_k\} = s(k \cdot T_S)$ kao rezultat IDFT($\{x_k\}$) dovoljni za potpunu rekonstrukciju OFDM signala $s(t)$? (40)

d) Na osnovu prethodnog, nacrtati principsku blok-šemu OFDM predajnika. Komentarisati zašto je ovo rešenje značajno pogodnije za implementaciju u odnosu na klasične FDM (*Frequency Division Multiplexing*) sisteme. (25)

Problem:

OFDM system (*Orthogonal Frequency Division Multiplexing*) transmits N digital signals $s_i(t), i = 1, 2, \ldots, N$, using N frequency carriers shifted by the frequency offset $\triangle f$ relative to each other. Complex baseband representation of signals $s_i(t)$ of i-th frequency channel is given as:

$$s_i(t) = x_i \cdot e^{2\pi j \cdot \triangle f \cdot i \cdot t}, \quad 0 \leq t \leq T, \tag{3}$$

where x_i are complex symbols of the modulation alphabet used (e.g. m-QAM, m-PSK, etc.) and T is the period or duration of the OFDM symbol. As a function of time, the OFDM symbol is given as the sum of all of N component signals:

$$s(t) = \sum_{i=1}^{N} s_i(t) = \sum_{i=1}^{N} x_i \cdot e^{2\pi j \cdot \triangle f \cdot i \cdot t}, \quad 0 \leq t \leq T. \tag{4}$$

a) Determine the minimal frequency separation $\triangle f$ of neighboring component signals $s_i(t)$ such that any pair of different component signals $s_i(t)$ and $s_k(t)$, $i \neq k$, are mutually orthogonal in the time-domain. (25)

b) Determine the set of N frequencies f_i selected as the carriers of component signals of the baseband OFDM signal $s(t)$ in such a way that minimizes the bandwidth of the OFDM signal $s(t)$. (10)

c) Let $\{x_k\}$, $k = 1, 2, \ldots$, be the stream of information symbols of the modulation alphabet of symbol duration $T_S = \frac{T}{N}$ at the input of the OFDM transmitter. During the OFDM symbol interval T OFDM transmitter accepts N information symbols at its input and produces the OFDM signal $s(t)$ at its output. Show that the time samples of the OFDM signal $s(t)$ sampled at the time instants $t = k \cdot T_S$ are Inverse Discrete Fourier Transform (IDFT) of the input information symbol stream $\{x_k\}$. Based on $\{s_k\} = s(k \cdot T_S)$ as the result of IDFT($\{x_k\}$), is it possible to fully reconstruct the original OFDM signal $s(t)$? (40)

d) Based on the previous development, sketch the basic scheme of the OFDM transmitter. Comment the practicality of developed implementation of the OFDM transmitter, particularly in comparison with the classic FDM (*Frequency Division Multiplexing*) systems. (25)

2 Statistička teorija telekomunikacija
Statistical Theory of Telecommunications

Zadatak:

Na slici 1 je dat prijemnik koji se sastoji od N korelatora. Za ulazni signal $x(t)$, na izlazu i-tog korelatora se dobija signal $y_i(t) = \int_0^T x(t)\psi_i(t)dt$. Set funkcija ψ_i, $i = 1, 2, \ldots, N$ je ortonormalan. Na ulazu u prijemnik je beli Gausov šum, nulte srednje vrednosti i spektralne gustine srednje snage $p_n = \frac{N_0}{2}$. Pokazati da je snaga šuma na izlazu svakog korelatora konačna i iste vrednosti, kao i da su komponente šuma međusobno statistički nezavisne.

Problem:

Figure 1 shows correlation receiver which consists of N correlators. Functions ψ_i, $i = 1, 2, \ldots, N$ are mutually orthonormal. Output of the each correlator is denoted by $y_i(t) = \int_0^T x(t)\psi_i(t)dt$, where $x(t)$ is an input signal. AWGN with zero mean and constant power spectral density $p_n = \frac{N_0}{2}$ is also present at receiver's input. Show that noise power at the output of each correlator is finite and of same value, and that noise components are mutually statistically independent.

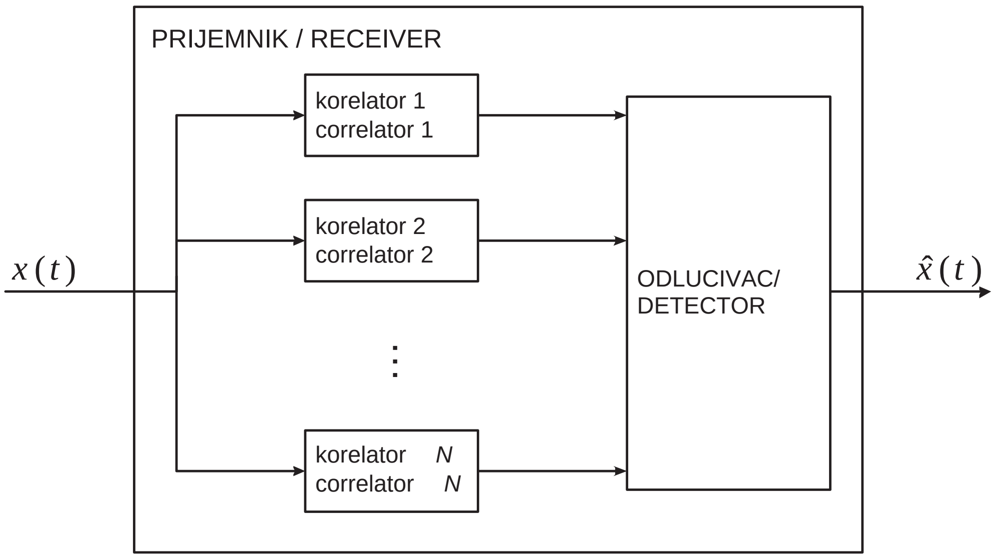

Figure 1: Korelacioni prijemnik / Correlation receiver

3 Teorija informacija
Information Theory

Zadatak:

U posudi se nalaze 4 istovetne kuglice obeležene brojevima 1, 2, 3 i 4. Na slučaj se izvlači prva kuglica i neka je njen broj X. Ta kuglica se vraća u posudu, a posle toga se ponovo izvlači kuglica. Ako joj je broj manji od X, vraća se u posudu i izvlačenje i vraćanje se ponavljaju sve dok se ne izvuče kuglica sa brojem Y, takvim da je $Y \geq X$. I ta kuglica se vraća u posudu i započinje nov niz izvlačenja i vraćanja dok se ne izvuče kuglica sa brojem Z, takvim da je $Z \leq Y$. Odrediti $I(X, Z)$.

Problem:

There are four identical balls in the bin, enumerated with numbers 1, 2, 3 and 4. Experiment starts by drawing one of the balls whose number is denoted by X. The ball is then returned to the bin, and another ball is drawn. If it's number is smaller than X, the ball is returned to the bin, and again another one is drawn. This action repeats until the ball with number Y such that $Y \geq X$ is drawn. This ball is also returned to the bin, and another series of drawing and returning of balls starts, which ends when a ball with number Z is drawn, such that $Z \leq Y$. Calculate the mutual information $I(X, Z)$.

4 Analogne modulacije
Analog Modulations

Zadatak:

Na ulaz prijemnika sa slike 2 dovodi se KAM (konvencionalno amplitudski modulisan) signal koji je modulisan test tonom učestanosti f_m. Indeks modulacije je m_0. Na ulazu prijemnika postoji i beli Gausov šum spektralne gustine srednje snage $p_n = \frac{N_0}{2}$. U prijemniku je idealan amplitudski detektor (blok DA na slici 2).

a) Odrediti odnos S/N u tačkama 1 i 2 (kod KAM modulisanog signala smatra se da korisni signal čine samo bočni opsezi)?

b) Ako je $p_n = 2 \cdot 10^{-20} \frac{\mathrm{W}}{\mathrm{Hz}}$, $U = 100\mathrm{mV}$ i $B = 10\mathrm{kHz}$, odrediti m_0 tako da odnos S/N u tački 2 bude 100dB. Koliki je tada odnos S/N u tački 1?

Problem:

Figure 2 shows conventional AM (CAM) signal receiver. At the receiver's input is present CAM signal modulated by test-tone (which is a cosinusoide) of frequency f_m. Modulation index is m_0. AWGN is also present at the receiver's input, with power spectral density $p_n = \frac{N_0}{2}$. Envelope detector (block denoted by DA in figure 2) can be considered as ideal.

a) Determine SNR in points 1 and 2 (only the power contained in the sidelobes of CAM signal is considered as effective).

b) For $p_n = 2 \cdot 10^{-20} \frac{\mathrm{W}}{\mathrm{Hz}}$, $U = 100\mathrm{mV}$ and $B = 10\mathrm{kHz}$, determine the value of m_0 such that SNR in point 2 is 100dB. What is the value of SNR in point 1?

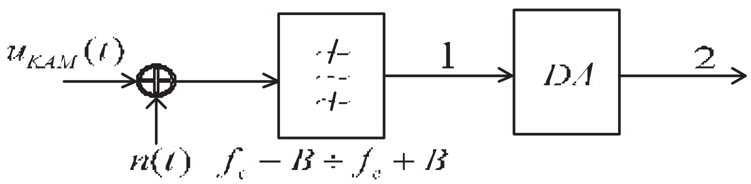

Figure 2: Prijemnik KAM signala / Conventional AM receiver

5 Digitalne telekomunikacije
Digital Telecommunications

Zadatak:

Ukoliko se za poredenje kvaliteta prenosa preko AWGN kanala koristi funkcija

$$C = P_b - 10^{-5} \cdot \eta$$

koja zavisi od verovatnoće greške po bitu (Pb) i spektralne efikasnosti modulacije (η), izabrati optimalnu modulacionu tehniku (tehniku koja minimizira C) iz ponuđenog skupa:

a) 16QAM sa pravugaonom konstelacijom,

b) nekoherentno detektovana ortogonalna 16FSK,

c) koherentno detektovana 16PSK.

Prilikom prenosa parametri pojedinih tehnika se podešavaju tako da odnos prosečne energije po bitu (Eb) i jednostrane spektralne snage šuma (N_0) bude konstantan i jednak 10dB. Mapiranje bita u simbole kod QAM i PSK modulacija zasnovano je na Gray-ovom kodovanju.

Problem:

Followung function is used to determine the quality of digital transmission over AWGN channel:

$$C = P_b - 10^{-5} \cdot \eta$$

where Pb is bit-error rate and η is spectral efficiency of transmission technique. Choose the optimal transmission technique (the one that minimizes C) among:

a) 16QAM with rectangular constellation,

b) non-coherently detected orthogonal 16FSK,

c) coherently detected 16PSK.

The ratio of average energy per bit (E_b) and one-sided power spectral density of AWGN (N_0) is equal 10dB for all above given techniques. Mapping of bits into symbols for QAM and PSK is based on Gray code.

1 Analiza signala i sistema
Signals and systems

Zadatak:

Problem:

2 Statistička teorija telekomunikacija
Statistical Theory of Telecommunications

Zadatak:

Na slici 1 je dat prijemnik koji se sastoji od N korelatora. Za ulazni signal $x(t)$, na izlazu i-tog korelatora se dobija signal $y_i(t) = \int_0^T x(t)\psi_i(t)dt$. Set funkcija ψ_i, $i = 1, 2, \ldots, N$ je ortonormalan. Na ulazu u prijemnik je beli Gausov šum, nulte srednje vrednosti i spektralne gustine srednje snage $p_n = \frac{N_0}{2}$. Pokazati da je snaga šuma na izlazu svakog korelatora konačna i iste vrednosti, kao i da su komponente šuma međusobno statistički nezavisne.

Problem:

Figure 1 shows correlation receiver which consists of N correlators. Functions ψ_i, $i = 1, 2, \ldots, N$ are mutually orthonormal. Output of the each correlator is denoted by $y_i(t) = \int_0^T x(t)\psi_i(t)dt$, where $x(t)$ is an input signal. AWGN with zero mean and constant power spectral density $p_n = \frac{N_0}{2}$ is also present at receiver's input. Show that noise power at the output of each correlator is finite and of same value, and that noise components are mutually statistically independent.

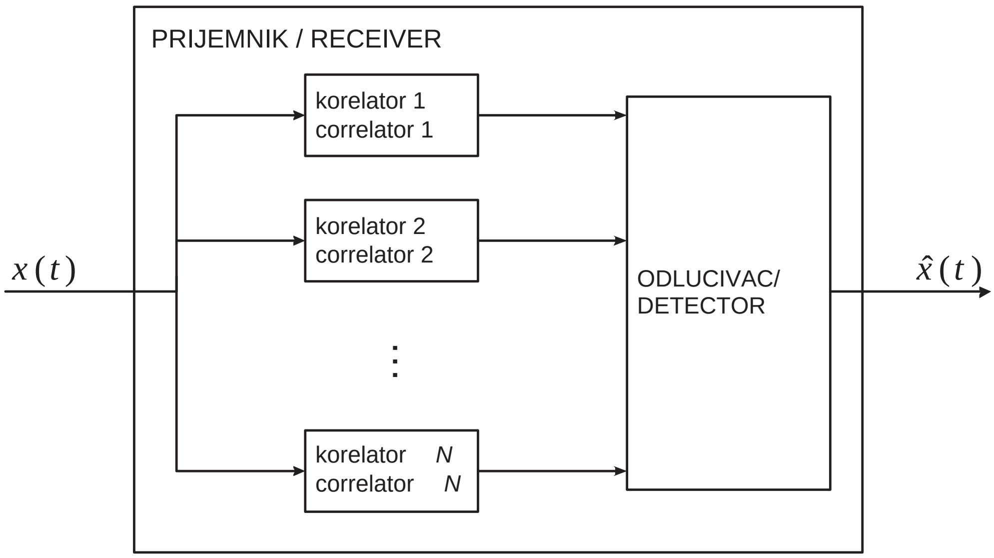

Figure 1: Korelacioni prijemnik / Correlation receiver

Rešenje:
Na izlazu svakog korelatora je slučajna promenljiva:

$$n_i = \int_0^T n(t)\psi_i(t)dt$$

Srednja vrednost n_i je:

$$\begin{aligned}\mathbf{E}\{n_i\} &= \mathbf{E}\{\int_0^T n(t)\psi_i(t)dt\} \\ &= \int_0^T \mathbf{E}\{n(t)\}\psi_i(t)dt = 0\end{aligned}$$

Varijansa n_i je:

$$\begin{aligned}\sigma_i^2 &= \mathbf{E}\{n_i^2\} - \mathbf{E}^2\{n_i\} = \mathbf{E}\{n_i^2\} \\ &= \mathbf{E}\{\int_0^T n(t)\psi_i(t)dt \int_0^T n(s)\psi_i(s)ds\} \\ &= \int_0^T \int_0^T \mathbf{E}\{n(t)n(s)\}\psi_i(t)\psi_i(s)dt\,ds \\ &= \int_0^T \int_0^T \mathcal{R}_n(t,s)\psi_i(t)\psi_i(s)dt\,ds \\ &= \int_0^T \int_0^T \mathcal{R}_n(t-s)\psi_i(t)\psi_i(s)dt\,ds \qquad (1)\end{aligned}$$

Za beli Gausov šum je:

$$\mathcal{R}_n(\tau) = \frac{N_0}{2}\delta(\tau) \qquad (2)$$

zamenom 2 u 1 dobijamo:

$$\begin{aligned}\sigma_i^2 &= \int_0^T \int_0^T \frac{N_0}{2}\delta(t-s)\psi_i(t)\psi_i(s)dt\,ds \\ &= \frac{N_0}{2}\int_0^T \psi_i^2(t)dt \\ &= \frac{N_0}{2} \qquad (3)\end{aligned}$$

Na osnovu jednačine 3 se vidi da je snaga na izlazu svakog korelatora ista i konačne vrednosti.

Ostalo je još da pokažemo da su komponente šuma koje su prošle kroz pojedine korelatore međsobno nezavisne. Pošto su komponente šuma takodje Gausovi procesi, dovoljno je pokazati da su one medjusobno nekorelisane, odnosno da je kovarijansa dve komponente šuma jednaka nuli. Na potpuno isti način kao i kod određivanja snage pojedinih komponenti, može se pokazati:

$$\mathbf{E}\{n_i n_j\} = \frac{N_0}{2}\int_0^T \psi_i(t)\psi_j(t)dt = 0$$

3 Teorija informacija Information Theory

Zadatak:

U posudi se nalaze 4 istovetne kuglice obeležene brojevima 1, 2, 3 i 4. Na slučaj se izvlači prva kuglica i neka je njen broj X. Ta kuglica se vraća u posudu, a posle toga se ponovo izvlači kuglica. Ako joj je broj manji od X, vraća se u posudu i izvlačenje i vraćanje se ponavljaju sve dok se ne izvuče kuglica sa brojem Y, takvim da je $Y \geq X$. I ta kuglica se vraća u posudu i započinje nov niz izvlačenja i vraćanja dok se ne izvuče kuglica sa brojem Z, takvim da je $Z \leq Y$. Odrediti $I(X, Z)$.

Problem:

There are four identical balls in the bin, enumerated with numbers 1, 2, 3 and 4. Experiment starts by drawing one of the balls whose number is denoted by X. The ball is then returned to the bin, and another ball is being drawn. If it's number is smaller than X, the ball is returned to the bin, and again another one is being drawn. This action repeats until the ball with number Y, such that $Y \geq X$, is drawn. That ball is also returned to the bin, and another series of drawing and returning of balls starts. The experiment ends when a ball with number Z is drawn, such that $Z \leq Y$. Calculate the mutual information $I(X, Z)$.

Rešenje:

Po definiciji međusobne informacije je

$$I(X, Z) = H(Z) - H(Z|X).$$

Da bi se odredilo $H(Z)$ i $H(Z|X)$, potrebno je odrediti verovatnoće $P[Z = z]$ i $P[Z = z|X = x]$. S obzirom da zbog načina izvlačenja kuglica važi $P[Z = z|X = x, Y = y] = P[Z = z|Y = y]$, dobija se

$$\begin{aligned} P[Z = z|X = x] &= \frac{P[X = x, Z = z]}{P[X = x]} = \\ &= \frac{\sum_{y=1}^{4} P[X = x, Y = y, Z = z]}{P[X = x]} = \\ &= \frac{\sum_{y=1}^{4} P[Z = z|X = x, Y = y] P[X = x, Y = y]}{P[X = x]} = \\ &= \sum_{y=1}^{4} P[Y = y|X = x] P[Z = z|Y = y] \end{aligned}$$

Neka matrice $A = [a_{i,j}]$, $B = [b_{i,j}]$ i $C = [c_{i,j}]$ imaju elemente

$$\begin{aligned} a_{i,j} &= P[Y = j|X = i], \\ b_{i,j} &= P[Z = j|Y = i], \\ c_{i,j} &= P[Z = j|X = i]. \end{aligned}$$

Na osnovu prethodne veze sledi da je

$$C = AB.$$

Iz definicija slučajnih promenljivih Y i Z dobija se

$$A = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix},$$

odakle je

$$C = \begin{bmatrix} \frac{25}{48} & \frac{13}{48} & \frac{7}{48} & \frac{1}{16} \\ \frac{13}{36} & \frac{13}{36} & \frac{7}{36} & \frac{1}{12} \\ \frac{7}{24} & \frac{7}{24} & \frac{7}{24} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

Kako je $P[X = x] = \frac{1}{4}$ i

$$\begin{aligned} P[Z = z] &= \sum_{x=1}^{4} P[X = x, Z = z] = \\ &= \sum_{x=1}^{4} P[Z = z|X = x]P[X = x] = \\ &= \frac{1}{4}\sum_{x=1}^{4} P[Z = z|X = x], \end{aligned}$$

dobija se

$$P[Z = 1] = \frac{205}{576}, \qquad P[Z = 2] = \frac{169}{576}, \qquad P[Z = 3] = \frac{127}{576}, \qquad P[Z = 4] = \frac{75}{576}.$$

Sada je

$$H(Z) = \frac{205}{576}\,\mathrm{ld}\,\frac{576}{205} + \frac{169}{576}\,\mathrm{ld}\,\frac{576}{169} + \frac{127}{576}\,\mathrm{ld}\,\frac{576}{127} + \frac{75}{576}\,\mathrm{ld}\,\frac{576}{75} \approx 1.913386112$$

i

$$\begin{aligned} H(Z|X = 1) &= \frac{25}{48}\,\mathrm{ld}\,\frac{48}{25} + \frac{13}{48}\,\mathrm{ld}\,\frac{48}{13} + \frac{7}{48}\,\mathrm{ld}\,\frac{48}{7} + \frac{1}{16}\,\mathrm{ld}\,16 \approx 1.655618896, \\ H(Z|X = 2) &= 2\cdot\frac{13}{36}\,\mathrm{ld}\,\frac{36}{13} + \frac{7}{36}\,\mathrm{ld}\,\frac{36}{7} + \frac{1}{12}\,\mathrm{ld}\,12 \approx 1.819430427, \\ H(Z|X = 3) &= 3\cdot\frac{7}{24}\,\mathrm{ld}\,\frac{24}{7} + \frac{1}{8}\,\mathrm{ld}\,8 \approx 1.930406632, \\ H(Z|X = 4) &= 4\cdot\frac{1}{4}\,\mathrm{ld}\,4 = 2, \end{aligned}$$

odnosno

$$H(Z|X) = \sum_{x=1}^{4} P[X = x]H(Z|X = x) = \frac{1}{4}\sum_{x=1}^{4} H(Z|X = x) \approx 1.851363989,$$

pa je konačno

$$I(X, Z) \approx 0.06202212362.$$