# MATHEMATICS II -May 2012 Kranevo, Bulgaria

1. Examine the convergence of the following series

$$\sum_{n=1}^{\infty} a_n, \quad a_n = \sqrt{n+a} - \sqrt[4]{n^2+n+b},$$

   where is $a, b \in \mathbb{R}$.

2. Find the area of border of the following region

$$V = \{(x,y,z) \in \mathbb{R}^3 : \ x+y \leq 1, \ z^2 \leq 2xy\}.$$

3. If $p \in \mathbb{R}$, calculate the given integral:

$$\int_0^{\infty} \frac{x^p dx}{(x^2+1)^2}.$$

# MATEMATIKA II -maj 2012 Kranevo, Bugarska

1. Ispitati konvegenciju reda

$$\sum_{n=1}^{\infty} a_n, \quad a_n = \sqrt{n+a} - \sqrt[4]{n^2+n+b},$$

   gde je $a, b \in \mathbb{R}$.

2. Izračunati površinu ruba oblasti

$$V = \{(x,y,z) \in \mathbb{R}^3 : \ x+y \leq 1, \ z^2 \leq 2xy\}.$$

3. Ako je $p \in \mathbb{R}$, izračunati integral:

$$\int_0^{\infty} \frac{x^p dx}{(x^2+1)^2}.$$

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**1.** Za ispitivanje konvegencije reda koristićemo uporedni kriterijum. Stoga:

$$a_n = \frac{(\sqrt{n+a})^2 - (\sqrt[4]{n^2+n+b})^2}{\sqrt{n+a} + \sqrt[4]{n^2+n+b}}$$

$$= \frac{(n+a) - \sqrt{n^2+n+b}}{\sqrt{n+a} + \sqrt[4]{n^2+n+b}} \cdot \frac{(n+a) + \sqrt{n^2+n+b}}{(n+a) + \sqrt{n^2+n+b}}$$

$$= \frac{(2a-1)n + a^2 - b}{(\sqrt{n+a} + \sqrt[4]{n^2+n+b})(n+a+\sqrt{n^2+n+b})}.$$

Ako je $a = \frac{1}{2}$ imamo $a_n \sim \frac{a^2-b}{2\sqrt{n}\cdot 2n} = \frac{a^2-b}{4n^{3/2}}$, pa iz konvergencije reda $\frac{a^2-b}{4}\sum\limits_{n=1}^{\infty}\frac{1}{n^{3/2}}$ sledi i konvergencija reda $\sum\limits_{n=1}^{\infty} a_n$.
Ako je $a \neq \frac{1}{2}$, tada je $a_n \sim \frac{(2a-1)n}{2\sqrt{n}\cdot 2n} = \frac{2a-1}{4n^{1/2}}$, pa iz divergencije reda $\frac{2a-1}{4}\sum\limits_{n=1}^{\infty}\frac{1}{n^{1/2}}$ sledi i divergencija reda $\sum\limits_{n=1}^{\infty} a_n$.

**2.** Oblast $V$ je simetrična u odnosu na $xOy$ ravan. Stoga ćemo posmatrati samo površinu gornje polovine oblasti. Ona se sastoji od površi $S_1 : z = \sqrt{2xy}$, $(x,y) \in D_1 = \{(x,y) \in \mathbb{R}^2 : \ x+y \leq 1,\ x,y \geq 0\} = \{(x,y) \in \mathbb{R}^2 : \ 0 \leq x \leq 1,\ 0 \leq y \leq 1-x\}$ i površi $S_2 : y = 1-x$, $(x,z) \in D_2 = \{(x,z) \in \mathbb{R}^2 : \ 0 \leq x \leq 1,\ 0 \leq z \leq \sqrt{2x(1-x)}\}$. Stoga je $p = 2(p_1 + p_2)$, gde su $p_i$ površine od $S_i$, $i = 1, 2$. Za prvu površ je $dS = \sqrt{1 + z_x^2 + z_y^2}dxdy = \sqrt{1 + (\frac{y}{\sqrt{2xy}})^2 + (\frac{y}{\sqrt{2xy}})^2}dxdy = \frac{x+y}{\sqrt{2xy}}dxdy$, dok za drugu $dS = \sqrt{1 + y_x^2 + y_z^2}dxdz = \sqrt{1 + (-1)^2 + 0^2}dxdz = \sqrt{2}dxdz$.

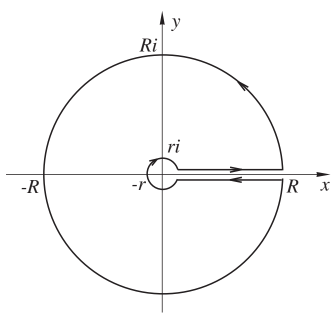
$p_1 = \iint\limits_{S_1} dS = \iint\limits_{D_1} \frac{x+y}{\sqrt{2xy}}dxdy = \frac{1}{\sqrt{2}}\iint\limits_{D_1}\left(x^{\frac{1}{2}}y^{-\frac{1}{2}} + x^{-\frac{1}{2}}y^{\frac{1}{2}}\right)dxdy = \frac{1}{\sqrt{2}}\int\limits_0^1 dx \int\limits_0^{1-x}(x^{\frac{1}{2}}y^{-\frac{1}{2}} + x^{-\frac{1}{2}}y^{\frac{1}{2}})dy =$
$\sqrt{2}\int\limits_0^1(x^{\frac{1}{2}}(1-x)^{\frac{1}{2}} + \frac{1}{3}x^{-\frac{1}{2}}(1-x)^{\frac{3}{2}})dx = \sqrt{2}(B(\frac{3}{2},\frac{3}{2}) + \frac{1}{3}B(\frac{1}{2},\frac{5}{2})) = \sqrt{2}(\frac{\Gamma^2(\frac{3}{2})}{\Gamma(3)} + \frac{1}{3}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{5}{2})}{\Gamma(3)}) = \sqrt{2}(\frac{(\frac{1}{2}\Gamma(\frac{1}{2}))^2}{2!} + \frac{1}{3}\frac{\Gamma(\frac{1}{2})\cdot\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})}{2!}) = \frac{\sqrt{2}}{4}\Gamma^2(\frac{1}{2}) = \frac{\pi\sqrt{2}}{4}$.

$p_2 = \iint\limits_{S_2} dS = \iint\limits_{D_2}\sqrt{2}dxdz = \sqrt{2}\int\limits_0^1 dx \int\limits_0^{\sqrt{2(x-x^2)}} dz = \sqrt{2}\int\limits_0^1\sqrt{2(x-x^2)}dx = 2\int\limits_0^1 x^{1/2}(1-x)^{1/2}dx = 2B\left(\frac{1}{2},\frac{1}{2}\right) = 2\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} = 2\frac{(\sqrt{\pi})^2}{1} = 2\pi$.

$$p = 2(\frac{\pi\sqrt{2}}{4} + 2\pi) = \frac{8+\sqrt{2}}{2}\pi.$$

**3.** Ispitajmo kada nesvojstveni integral konvergira.
$\int\limits_0^1 \frac{x^p}{(x^2+1)^2}dx \sim \int\limits_0^1 \frac{1}{x^{-p}}dx$ konvergira za $-p < 1$, tj. $p > -1$.
$\int\limits_1^{\infty} \frac{x^p}{(x^2+1)^2}dx \sim \int\limits_1^{\infty} \frac{1}{x^{4-p}}dx$ konvergira za $4 - p > 1$, tj. $p < 3$.
Zbog $\int\limits_0^{\infty}\frac{x^p dx}{(x^2+1)^2} = \int\limits_0^1\frac{x^p dx}{(x^2+1)^2} + \int\limits_1^{\infty}\frac{x^p dx}{(x^2+1)^2}$ imamo da integral konvergira za $p \in (-1, 3)$.

Razmatrajmo slučaj kada $p \in (-1,3) \setminus \{0,1,2\}$. U tu svrhu posmatramo kompleksni integral $\int\limits_L f(z)dz$, $f(z) = \frac{z^p}{(z^2+1)^2}$, gde je kontura $L$ data na slici.




Uzmimo onu granu funkcije $z^p = e^{p\ln z} = e^{p(\ln|z| + i\arg z + 2k\pi i)}$ za koju unutar date konture $L$ važi $\operatorname{Arg} z = \arg z + 2k\pi \in (0, 2\pi)$, tj. $\arg z \in (0, 2\pi)$. U tom slučaju će biti i

$z = x \in \mathbb{R}^+ \Rightarrow z^p = e^{p(\ln x + i\cdot 0)} = e^{p\ln x} = x^p$, $\quad z = xe^{i2\pi}, x \in \mathbb{R}^+ \Rightarrow z^p = e^{p(\ln x + i2\pi)} = e^{p\ln x}e^{2p\pi i} = x^p e^{2p\pi i}$,

$(z^2+1)^2 = (z-i)^2(z+i)^2 = 0 \Leftrightarrow z_{1/2} = \pm i \in \operatorname{int} L$. Odatle, sledi da su $z_{1/2}$ polovi drugog reda funkcije $f(z) = \frac{z^p}{(z^2+1)^2}$, pa je

$I=\int\limits_L f(z)dz=2\pi i\,(\operatorname*{Res}_{z=z_1} f(z)+\operatorname*{Res}_{z=z_2} f(z))=2\pi i\cdot\Big(\lim_{z\to i}\big((z-i)^2\frac{z^p}{(z^2+1)^2}\big)'+\lim_{z\to -i}\big((z+i)^2\frac{z^p}{(z^2+1)^2}\big)'\Big)$
$=2\pi i\cdot\Big(\lim_{z\to i}\big(\frac{e^{p\ln z}}{(z+i)^2}\big)'+\lim_{z\to -i}\big(\frac{e^{p\ln z}}{(z-i)^2}\big)'\Big)=2\pi i\cdot\Big(\lim_{z\to i}\big(\frac{e^{p\ln z}\frac{p}{z}(z+i)^2-2(z+i)e^{p\ln z}}{(z+i)^4}\big)+\lim_{z\to -i}\big(\frac{e^{p\ln z}\frac{p}{z}(z-i)^2-2(z-i)e^{p\ln z}}{(z-i)^4}\big)\Big)$
$=2\pi i\Big(\frac{e^{pi\pi/2}\frac{p}{i}(i+i)^2-2(i+i)e^{pi\pi/2}}{(i+i)^4}+\frac{e^{3pi\pi/2}\frac{p}{-i}(-i-i)^2-2(-i-i)e^{3pi\pi/2}}{(-i-i)^4})\Big)=2\pi i\Big(e^{pi\pi/2}\frac{p-1}{4}i+e^{3pi\pi/2}\frac{1-p}{4}i\Big)$
$=\frac{1-p}{2}\pi e^{pi\pi}(e^{-pi\pi/2}-e^{pi\pi/2}).$

Sa druge strane je
$I=\int\limits_r^R\frac{x^p}{(x^2+1)^2}dx+\int\limits_0^{2\pi}\frac{R^pe^{pit}}{(R^2e^{2it}+1)^2}Rie^{it}dt+\int\limits_R^r\frac{x^pe^{pi2\pi}}{(x^2e^{4i\pi}+1)^2}e^{2i\pi}dx+\int\limits_{2\pi}^0\frac{r^pe^{pit}}{(r^2e^{2it}+1)^2}rie^{it}dt=I_1+I_2+I_3+I_4.$

$|I_2|\le\int\limits_0^{2\pi}\frac{|R^pe^{pit}|}{||R^2e^{2it}|-1|^2}|Rie^{it}|dt=\int\limits_0^{2\pi}\frac{R^p}{(R^2-1)^2}Rdt=\frac{2\pi R^{1+p}}{(R^2-1)^2}\ \Rightarrow\ \lim_{R\to\infty}I_2=0$, jer $\frac{2\pi R^{1+p}}{(R^2-1)^2}\sim 2\pi R^{p-3}\to 0,\ R\to\infty,$ $p<3$.

Slično, $|I_4|\le\int\limits_0^{2\pi}\frac{|r^pe^{pit}|}{||r^2e^{2it}|-1|^2}|rie^{it}|dt=\int\limits_0^{2\pi}\frac{r^p}{(1-r^2)^2}rdt=\frac{2\pi r^{1+p}}{(1-r^2)^2}\ \Rightarrow\ \lim_{r\to 0}I_4=0$, jer $\frac{2\pi r^{1+p}}{(1-r^2)^2}\to 0,\ r\to 0,\ p>-1.$

Dakle, kada $r\to 0,\ R\to\infty$, tada je $(1-e^{2pi\pi})\int\limits_0^\infty\frac{x^pdx}{(x^2+1)^2}=\frac{1-p}{2}\pi e^{pi\pi}(e^{-pi\pi/2}-e^{pi\pi/2})$, tj.

$$\int\limits_0^\infty\frac{x^pdx}{(x^2+1)^2}=\frac{1}{e^{pi\pi}(e^{-pi\pi}-e^{p\pi i})}\cdot\frac{1-p}{2}\pi e^{pi\pi}(e^{-pi\pi/2}-e^{pi\pi/2})$$

$$=\frac{1}{(e^{-pi\pi/2}-e^{p\pi i/2})(e^{-pi\pi/2}+e^{p\pi i/2})}\cdot\frac{1-p}{2}\pi(e^{-pi\pi/2}-e^{pi\pi/2})=\frac{(1-p)\pi}{4\cos\frac{p\pi}{2}}.$$

Za $p\in\{0,1,2\}$, $z=0$ nije tačka grananja. Ako je $p=0$ ili $p=2$ za $L$ uzimamo polukružnu putanju sa centrom u 0, prečnika $2R$ duž realne ose. Tada funkcija $f(z)=\frac{z^p}{(z^2+1)^2}$ ima unutar konture samo singularitet $z=i$ pol drugog reda, pa je $I=\int\limits_L f(z)dz=\int\limits_{-R}^R\frac{x^p}{(x^2+1)^2}dx+\int\limits_0^\pi\frac{R^pe^{pit}}{(R^2e^{2it}+1)^2}Rie^{it}dt=2\pi i\operatorname*{Res}_{z=i}f(z).$

Podintegralna funkcija je parna te je $\int\limits_{-R}^R\frac{x^p}{(x^2+1)^2}dx=2\int\limits_0^R\frac{x^p}{(x^2+1)^2}dx$, pa uzimajući da $R\to\infty$, slično, kao napred imamo da je $\lim_{R\to\infty}\int\limits_0^\pi\frac{R^pe^{pit}}{(R^2e^{2it}+1)^2}Rie^{it}dt=0$. Tako imamo

$$\int\limits_0^\infty\frac{x^pdx}{(x^2+1)^2}=\pi i\operatorname*{Res}_{z=i}f(z)=\pi i\lim_{z\to i}\big(\frac{z^p}{(z+i)^2}\big)'=\pi i\cdot\frac{1-p}{4}i^{p-1}=\frac{1-p}{4}\pi i^p=\frac{1-p}{4}\pi(-1)^{1-p}.$$

Ako je $p=1$ onda je integral jednak $\int\limits_0^\infty\frac{xdx}{(x^2+1)^2}=\frac{1}{2}\int\limits_1^\infty\frac{dt}{t^2}=\lim_{T\to\infty}\frac{1}{2}(-\frac{1}{T}+1)=\frac{1}{2}.$