MATHEMATICS II -May 2011 Kopaonik

1. Examine the convergence of the following series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n!}\,(1+a^{2n}b^2)},$$

where is $a, b \in \mathbb{R}$.

2. Find volume of the following region

$$V = \{(x,y,z) \in \mathbb{R}^{+3} : \left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)^3 \le \ln \frac{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}}{\frac{x}{a}+\frac{y}{b}}\} \quad (a,b,c>0).$$

3. a) Determine analytical function $f(z) = u(x,y) + iv(x,y)$ that is

$$v(x,y) = \frac{1-y}{(x-1)^2+(y-1)^2}$$

   and $f(1+2i) = -i$.

   b) Determine $g(G)$, if $g(z) = e^{\pi(z-1)f(z)}$ and $G = \{z \in \mathbb{C} : |z-1| < 1, |z-i| > 1, \operatorname{Re} z < 1\}$.

MATEMATIKA II -maj 2011 Kopaonik

1. Ispitati konvegenciju reda

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n!}\,(1+a^{2n}b^2)},$$

gde je $a, b \in \mathbb{R}$.

2. Naći zapreminu oblasti

$$V = \{(x,y,z) \in \mathbb{R}^{+3} : \left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)^3 \le \ln \frac{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}}{\frac{x}{a}+\frac{y}{b}}\} \quad (a,b,c>0).$$

3. a) Odrediti analitičku funkciju $f(z)$ tako da je njen imaginarni deo funkcija

$$v(x,y) = \frac{1-y}{(x-1)^2+(y-1)^2}$$

   i da je $f(1+2i) = -i$.

   b) Za tako određenu funkciju $f(z)$, funkcijom $g(z) = e^{\pi(z-1)f(z)}$ preslikati oblast $\{z \in \mathbb{C} : |z-1| < 1, |z-i| > 1, \operatorname{Re} z < 1\}$.

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1. Za ispitivanje konvergencije reda $\sum\limits_{n=1}^{\infty} a_n,\ a_n = \frac{1}{\sqrt[n]{n!}\,(1+a^{2n}b^2)}$, koristićemo uporedni kriterijum.

Iz Stirlingove formule $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$, imamo $\sqrt[n]{n!} \sim \sqrt[n]{\sqrt{2\pi}} \cdot \sqrt{\sqrt[n]{n}} \cdot \frac{n}{e}$, odakle zbog $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1\ (a>0),\ \lim\limits_{n\to\infty} \sqrt[n]{n} = 1$, sledi $\sqrt[n]{n!} \sim \frac{1}{e}n$.

Za $b = 0$ je $a_n = \frac{1}{\sqrt[n]{n!}} \sim \frac{e}{n}$, te iz divergencije reda $e\sum\limits_{n=1}^{\infty} \frac{1}{n}$ sledi i divergencija reda $\sum\limits_{n=1}^{\infty} a_n$.

Ako je $|a| \le 1$, tada je $a_n \sim \frac{e}{n} \cdot \frac{1}{1+a^{2n}b^2} \sim \frac{e}{n}$, pa iz divergencije reda $e\sum\limits_{n=1}^{\infty} \frac{1}{n}$ sledi i divergencija reda $\sum\limits_{n=1}^{\infty} a_n$.

Ukoliko je $b \neq 0$ i $|a| > 1$, važi $a_n \sim \frac{e}{n} \cdot \frac{1}{b^2}(\frac{1}{a^2})^n$.

Kako je $\frac{e}{n} \cdot \frac{1}{b^2}(\frac{1}{a^2})^n \le \frac{e}{b^2}(\frac{1}{a^2})^n,\ n \in \mathbb{N}$ i geometrijski red $\frac{e}{b^2}\sum\limits_{n=1}^{\infty}(\frac{1}{a^2})^n$ konvergira jer je $|\frac{1}{a^2}| < 1$, pa odatle sledi i konvergencija reda $\sum\limits_{n=1}^{\infty} a_n$.

2. $x = a\rho\cos^{\alpha}\varphi\sin^{\beta}\theta,\ y = b\rho\sin^{\alpha}\varphi\sin^{\beta}\theta,\ z = c\rho\cos^{\beta}\theta,\ \theta \in [0,\pi],\ \varphi \in [0,2\pi),\ \rho \ge 0$, $J = \alpha\beta abc\rho^2\sin^{\alpha-1}\varphi\cos^{\alpha-1}\varphi\sin^{2\beta-1}\theta\cos^{\beta-1}\theta$.

$x,y,z > 0 \quad\Rightarrow\quad \theta \in (0,\frac{\pi}{2}),\ \varphi \in (0,\frac{\pi}{2}),\quad (\frac{x}{a}+\frac{y}{b}+\frac{z}{c})^3 = \ln\frac{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}}{\frac{x}{a}+\frac{y}{b}} \Rightarrow \rho = -\sqrt[3]{2\ln(\sin\theta)}$ $\Rightarrow \rho \in (0, -\sqrt[3]{2\ln(\sin\theta)});\quad J = 4abc\rho^2\sin\varphi\cos\varphi\sin^3\theta\cos\theta$.

$v = \iiint\limits_V dxdydz = \iiint\limits_{V'} |J|d\rho d\varphi d\theta = \int\limits_0^{\pi/2}\int\limits_0^{\pi/2}\int\limits_0^{-\sqrt[3]{2\ln(\sin\theta)}} 4abc\rho^2\sin\varphi\cos\varphi\sin^3\theta\cos\theta =$ $4abc\int\limits_0^{\pi/2}\sin\varphi\cos\varphi d\varphi\int\limits_0^{\pi/2}\sin^3\theta\cos\theta(\int\limits_0^{-\sqrt[3]{2\ln(\sin\theta)}}\rho^2 d\rho)d\theta = 4abc\cdot\frac{1}{2}\sin^2\varphi|_0^{\pi/2}\int\limits_0^{\pi/2}\sin^3\theta\cos\theta\cdot$ $\frac{1}{3}(-\sqrt[3]{2\ln(\sin\theta)})^3 d\theta = -\frac{4}{3}abc\int\limits_0^{\pi/2}\sin^3\theta\cos\theta\ln(\sin\theta)d\theta = -\frac{4}{3}abc\int\limits_0^1 t^3\ln t dt = -\frac{4}{3}abc\lim\limits_{\varepsilon\to0^+}\int\limits_{\varepsilon}^1 t^3\ln t dt$.

Parcijalnom integracijom za $m = \ln t,\ dm = \frac{1}{t}dt,\ dn = t^3dt,\ n = \frac{1}{4}t^4$ je $\int t^3\ln t dt = \frac{t^4}{4}\ln t - \frac{1}{4}\int t^3 dt = \frac{1}{4}t^4\ln t - \frac{1}{16}t^4$.

$v = -\frac{4}{3}abc[-\frac{1}{16} - \lim\limits_{\varepsilon\to0^+}(\frac{1}{4}\varepsilon^4\ln\varepsilon - \frac{1}{16}\varepsilon^4)] = -\frac{4}{3}abc[-\frac{1}{16} - \frac{1}{4}\lim\limits_{\varepsilon\to0^+}\frac{\ln\varepsilon}{\varepsilon^{-4}} - \frac{1}{16}\cdot 0] = -\frac{4}{3}abc[-\frac{1}{16} - \frac{1}{4}\lim\limits_{\varepsilon\to0^+}\frac{\varepsilon^{-1}}{-4\varepsilon^{-5}}] = \frac{abc}{12}$.

3. a) Iz Koši-Rimanovih jednačina sledi $u_y = -v_x = -\frac{2(x-1)(y-1)}{((x-1)^2+(y-1)^2)^2}$, pa je

$u = -(x-1)\int\frac{2(y-1)}{((x-1)^2+(y-1)^2)^2}dy = \frac{x-1}{(x-1)^2+(y-1)^2} + \varphi(x)$.

Takođe je

$u_x = v_y \Leftrightarrow \frac{(x-1)^2+(y-1)^2-2(x-1)^2}{((x-1)^2+(y-1)^2)^2} + \varphi'(x) = \frac{-(x-1)^2-(y-1)^2+2(y-1)^2}{((x-1)^2+(y-1)^2)^2}$,
pa je $\varphi'(x) = 0,\ \varphi(x) = c$, odnosno

$$f(z) = f(x+yi) = \frac{x-1}{(x-1)^2+(y-1)^2} + c + i\frac{1-y}{(x-1)^2+(y-1)^2}.$$

Iz $f(1+2i) = c - i = -i$ je $c = 0$ i

$f(x+yi) = \frac{(x-1)-i(y-1)}{(x-1)^2+(y-1)^2} = \frac{\overline{z-1-i}}{|z-1-i|^2} = \frac{1}{z-1-i}$.

b) $w = e^{\pi(z-1)f(z)} = e^{\pi\frac{z-1}{z-1-i}} = e^{\pi(1+i/(z-1-i))}$.

$w_1 = z - 1 - i$, $w_2 = \frac{1}{w_1}$, $w_3 = iw_2$, $w_4 = 1 + w_3$, $w_5 = \pi w_4$, $w = e^{w_5}$.

Data oblast se preslikala na $\{w \in \mathbb{C} : \arg w \in (\frac{3\pi}{2}, 2\pi), |w| < e^{\frac{\pi}{2}}\}$.