MATHEMATICS II -May 2010 Čanj

1. Examine the convergence and find the sum of the following series

$$\sum_{n=0}^{\infty}(-1)^{\frac{n(n-1)}{2}}a^n, \quad a \in \mathbb{R}.$$

2. Find the volume of the following region

$$V = \{(x,y,z) \in \mathbb{R}^3 : \ (x^2+y^2)^3 + z^6 \leq a^3xyz\} \quad (a>0).$$

3. If $|a| \neq 1$, $|b| > 1$, $ab \neq 1$, $a, b \in \mathbb{R}$, calculate the given integral:

$$I = \int_0^{\pi} \frac{\sin^2 t}{(1-2a\cos t + a^2)(1-2b\cos t+b^2)}dt.$$

MATEMATIKA II -maj 2010 Čanj

1. Ispitati konvergenciju i naći sumu reda

$$\sum_{n=0}^{\infty}(-1)^{\frac{n(n-1)}{2}}a^n, \quad a \in \mathbb{R}.$$

2. Naći zapreminu oblasti:

$$V = \{(x,y,z) \in \mathbb{R}^3 : \ (x^2+y^2)^3 + z^6 \leq a^3xyz\} \quad (a>0).$$

3. Ako je $|a| \neq 1$, $|b| > 1$, $ab \neq 1$, $a, b \in \mathbb{R}$, izračunati integral:

$$I = \int_0^{\pi} \frac{\sin^2 t}{(1-2a\cos t + a^2)(1-2b\cos t+b^2)}dt.$$

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1. Ako je $a_n = (-1)^{\frac{n(n-1)}{2}} a^n$, tada je $\sum\limits_{n=0}^{\infty} |a_n| = \sum\limits_{n=0}^{\infty} a^n$, te red $\sum\limits_{n=0}^{\infty} a_n$, kao geometrijski red, apsolutno konvergira (pa time i obično) za $|a| < 1$.

Za $|a| \geq 1$ imamo $\lim\limits_{n\to\infty} |a_n| \neq 0$, pa je $\lim\limits_{n\to\infty} a_n \neq 0$, te u tom slučaju red $\sum\limits_{n=0}^{\infty} a_n$ divergira.

Zbog apsolutne konvergencije reda redosled sabiranja nije bitan te grupisanjem članova reda dobijamo da je suma s reda $\sum\limits_{n=0}^{\infty} a_n$ za $|a| < 1$ jednaka:

$s = 1 + a - a^2 - a^3 + a^4 + a^5 - ... = (1+a) - a^2(1+a) + a^4(1+a) - a^6(1+a) + ... = (1+a)(1 - a^2 + a^4 - a^6 + ...) = (1+a) \cdot \frac{1}{1-(-a^2)} = \frac{1+a}{1+a^2}$.

2. Uvešćemo uopštene sferne koordinate $x = ar\cos^{\alpha}\varphi\sin^{\beta}\theta$, $y = br\sin^{\alpha}\varphi\sin^{\beta}\theta$, $z = cr\cos^{\beta}\theta$, $\alpha,\beta \in \mathbb{Q}\setminus\{0\}$, $a > 0$, $b > 0$, $c > 0$, $(r,\varphi,\theta) \in D_{r\varphi\theta} \subseteq [0,\infty)\times[0,2\pi]\times[0,\pi]$, čiji je Jakobjan $J(r,\varphi,\theta) = -\alpha\beta abcr^2\sin^{\alpha-1}\varphi\cos^{\alpha-1}\varphi\sin^{2\beta-1}\theta\cos^{\beta-1}\theta$, odnosno za

$$x = r\cos\varphi\sin^{\frac{1}{3}}\theta,\ y = r\sin\varphi\sin^{\frac{1}{3}}\theta,\ z = r\cos^{\frac{1}{3}}\theta,\ J = -\frac{1}{3}r^2\sin^{-\frac{1}{3}}\theta\cos^{-\frac{2}{3}}\theta.$$

Površ prolazi kroz koordinatni početak i mora zadovoljavati $xyz \geq 0$. Uvrštavajući umesto x,y,z respektivno vrednosti $x,-y,-z$; $-x,y,-z$ i $-x,-y,z$ u jednačinu površi vidimo da se "telo" koje ona ograničava sastoji od 4 podudarna dela, te je stoga dovoljno posmatrati I oktant. Stoga uzimamo $\varphi \in [0,\frac{\pi}{2}]$, $\theta \in [0,\frac{\pi}{2}]$, $r \in [0, a\sqrt[3]{\cos\varphi\sin\varphi\sin^{\frac{2}{3}}\theta\cos^{\frac{1}{3}}\theta})$.

$v = 4\iiint\limits_{V_1} dxdydz = 4\iiint\limits_{V_1'} |J|drd\varphi d\theta = \frac{4}{3}\int\limits_0^{\pi/2}\sin^{-\frac{1}{3}}\theta\cos^{-\frac{2}{3}}\theta d\theta\int\limits_0^{\pi/2}d\varphi\int\limits_0^{a\sqrt[3]{\cos\varphi\sin\varphi\sin^{\frac{2}{3}}\theta\cos^{\frac{1}{3}}\theta}} r^2dr = \frac{4a^3}{9}\int\limits_0^{\pi/2}\sin^{\frac{1}{3}}\theta\cos^{-\frac{1}{3}}\theta d\theta\int\limits_0^{\pi/2}\sin\varphi\cos\varphi d\varphi = \frac{4a^3}{9}\cdot\frac{1}{2}B(\frac{2}{3},\frac{1}{3})\cdot\frac{1}{2}B(1,1) = \frac{a^3}{9}\frac{\Gamma(\frac{2}{3})\Gamma(\frac{1}{3})}{\Gamma(1)} = \frac{a^3}{9}\frac{\frac{\pi}{\sin\frac{\pi}{3}}}{1} = \frac{2a^3\pi}{27}$.

Koristili smo osobine: $\Gamma(1) = 1$; $\Gamma(p)\cdot\Gamma(1-p) = \frac{\pi}{\sin p\pi}$, $0 < p < 1$; $B(1,1) = \int\limits_0^1 t^{1-1}(1-t)^{1-1}dt = 1$; $B(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$; $B(m,n) = 2\int\limits_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\cdot\cos^{2n-1}\theta d\theta$.

3. Primetimo da je npr. $1 - 2a\cos t + a^2 = 0 \Leftrightarrow a = \cos t \pm \sqrt{\cos^2 t - 1} \in \mathbb{R} \Leftrightarrow \cos t = \pm 1 \Leftrightarrow a = \pm 1$, pa kako je $|a| \neq 1$, $|b| > 1$, stoga dati integral nije nesvojstven.

Zbog parnosti podintegralne funkcije, uvodeći smenu $z = e^{ti}$, $\sin t = \frac{z^2-1}{2zi}$, $\cos t = \frac{z^2+1}{2z}$, $dt = \frac{dz}{iz}$, zadati integral je jednak

$$I = \frac{1}{2}\int\limits_0^{2\pi}\frac{\sin^2 t}{(1-2a\cos t+a^2)(1-2b\cos t+b^2)}dt = \frac{1}{2}\int\limits_c\frac{(\frac{z^2-1}{2zi})^2}{(1-2a\frac{z^2+1}{2z}+a^2)(1-2b\frac{z^2+1}{2z}+b^2)}\frac{dz}{iz}$$

$$= \frac{i}{8}\int\limits_c\frac{(z^2-1)^2}{z(az^2-(a^2+1)z+a)(bz^2-(b^2+1)z+b)}dz.$$

$az^2 - (a^2+1)z + a = 0 \Leftrightarrow z = a \vee z = \frac{1}{a}$.

1) $|a| > 1$, $|b| > 1$, $a \neq b$:

$z = 0$ pol prvog reda $\operatorname*{Res}\limits_{z=0} f(z) = \lim\limits_{z\to 0}(\frac{(z^2-1)^2}{(az^2-(a^2+1)z+a)(bz^2-(b^2+1)z+b)}) = \frac{1}{ab}$;

$z = \frac{1}{a}$ pol prvog reda $\operatorname*{Res}\limits_{z=\frac{1}{a}} f(z) = \lim\limits_{z\to\frac{1}{a}}(\frac{(z^2-1)^2}{z\cdot a(z-a)b(z-b)(z-\frac{1}{b})}) = \frac{1-a^2}{a(1-ab)(b-a)}$;

$z=\frac{1}{b}$ pol prvog reda $\operatorname*{Res}_{z=\frac{1}{b}} f(z)=\lim_{z\to\frac{1}{b}}\left(\frac{(z^2-1)^2}{z\cdot a(z-a)(z-\frac{1}{a})b(z-b)}\right)=\frac{1-b^2}{b(1-ab)(a-b)}$;

$I=\frac{i}{8}\cdot 2\pi i\sum \operatorname{Res} f(z)=-\frac{\pi}{4}\left(\frac{1}{ab}+\frac{1-a^2}{a(1-ab)(b-a)}+\frac{1-b^2}{b(1-ab)(a-b)}\right)=\frac{\pi}{2ab(ab-1)}$.

2) $|a|>1$, $|b|>1$, $a=b$:

$z=0$ pol prvog reda $\operatorname*{Res}_{z=0} f(z)=\frac{1}{a^2}$;

$z=\frac{1}{a}$ pol drugog reda $\operatorname*{Res}_{z=\frac{1}{a}} f(z)=\lim_{z\to\frac{1}{a}}\left((z-\frac{1}{a})^2\cdot\frac{(z^2-1)^2}{z\cdot a^2(z-a)^2(z-\frac{1}{a})^2}\right)'=$

$\frac{1}{a^2}\lim_{z\to\frac{1}{a}}\frac{2(z^2-1)\cdot 2z\cdot z(z-a)^2-(3z^2-4az+a^2)(z^2-1)^2}{z^2(z-a)^4}=\frac{1+a^2}{a^2(1-a^2)}$;

$I=\frac{i}{8}\cdot 2\pi i\sum \operatorname{Res} f(z)=-\frac{\pi}{4}\left(\frac{1}{a^2}+\frac{1+a^2}{a^2(1-a^2)}\right)=\frac{\pi}{2a^2(a^2-1)}$.

3) $0<|a|<1$, $|b|>1$:

$z=a$ pol prvog reda $\operatorname*{Res}_{z=a} f(z)=\lim_{z\to a}\left(\frac{(z^2-1)^2}{z\cdot a(z-\frac{1}{a})b(z-b)(z-\frac{1}{b})}\right)=\frac{a^2-1}{a(a-b)(ab-1)}$;

$I=\frac{i}{8}\cdot 2\pi i\sum \operatorname{Res} f(z)=-\frac{\pi}{4}\left(\frac{1}{ab}+\frac{a^2-1}{a(a-b)(ab-1)}+\frac{1-b^2}{b(1-ab)(a-b)}\right)=\frac{\pi}{2b(b-a)}$.

4) $a=0$, $|b|>1$:

$z=0$ pol drugog reda $\operatorname*{Res}_{z=0} f(z)=\lim_{z\to 0}\left(z^2\cdot\frac{(z^2-1)^2}{-z^2(bz^2-(b^2+1)z+b)}\right)'$

$=-\lim_{z\to 0}\frac{4z(z^2-1)\cdot(bz^2-(b^2+1)z+b)-(z^2-1)^2\cdot(2bz-b^2-1)}{(bz^2-(b^2+1)z+b)^2}=-\frac{1+b^2}{b^2}$;

$z=\frac{1}{b}$ pol prvog reda $\operatorname*{Res}_{z=\frac{1}{b}} f(z)=\frac{b^2-1}{b^2}$;

$I=\frac{i}{8}\cdot 2\pi i\sum \operatorname{Res} f(z)=-\frac{\pi}{4}\left(-\frac{1+b^2}{b^2}+\frac{b^2-1}{b^2}\right)=\frac{\pi}{2b^2}$.