MATHEMATICS II -May 2009 Budva

1. Using the Maclaurin's series of the function $f(x) = \arcsin x$, find the sum:

$$\sum_{n=0}^{\infty} \binom{2n+1}{2} \frac{(2n-1)!!}{(2n+1)(2n)!!} z^{2n-1}, \qquad z \in (-1,1).$$

2. Find the volume of region

$$V = \{(x,y,z) \in \mathbb{R}^3 : \ 0 \le z \le 4 - \sqrt{x^2+y^2}, \ 2x \le x^2+y^2 \le 4x\}.$$

3. Map the region $G = \{z \in \mathbb{C} : \operatorname{Re} z < 0, \ 0 < \arg z < \frac{\pi}{n}\}$ $(n \in \mathbb{N}, \ n \ge 3)$ by given function

$$f(z) = \frac{i(1-(e^z)^n)^2 - (1+(e^z)^n)^2}{i(1-(e^z)^n)^2 + (1+(e^z)^n)^2}.$$

MATEMATIKA II -maj 2009 Budva

1. Koristeći razvoj funkcije $f(x) = \arcsin x$ u stepeni red u okolini tačke $x = 0$, naći sumu reda

$$\sum_{n=0}^{\infty} \binom{2n+1}{2} \frac{(2n-1)!!}{(2n+1)(2n)!!} z^{2n-1}, \qquad z \in (-1,1).$$

2. Izračunati zapreminu oblasti

$$V = \{(x,y,z) \in \mathbb{R}^3 : \ 0 \le z \le 4 - \sqrt{x^2+y^2}, \ 2x \le x^2+y^2 \le 4x\}.$$

3. Preslikati oblast $G = \{z \in \mathbb{C} : \operatorname{Re} z < 0, \ 0 < \arg z < \frac{\pi}{n}\}$ $(n \in \mathbb{N}, \ n \ge 3)$ funkcijom

$$f(z) = \frac{i(1-(e^z)^n)^2 - (1+(e^z)^n)^2}{i(1-(e^z)^n)^2 + (1+(e^z)^n)^2}.$$

Mentor takmičenja: Nebojša M. Ralević

Rešenje:

1. Za $|x| < 1$ imamo

$\arcsin x = \sum\limits_{n=0}^{\infty} \frac{(2n-1)!!}{(2n+1)(2n)!!} (x - z + z)^{2n+1} = \sum\limits_{n=0}^{\infty} \sum\limits_{k=0}^{2n+1} \binom{2n+1}{k} \frac{(2n-1)!!}{(2n+1)(2n)!!} (x-z)^k z^{2n+1-k}$

$= \sum\limits_{k=0}^{\infty} \Big(\sum\limits_{n=[\frac{k}{2}]}^{\infty} \binom{2n+1}{k} \frac{(2n-1)!!}{(2n+1)(2n)!!} z^{2n+1-k} \Big) (x-z)^k.$

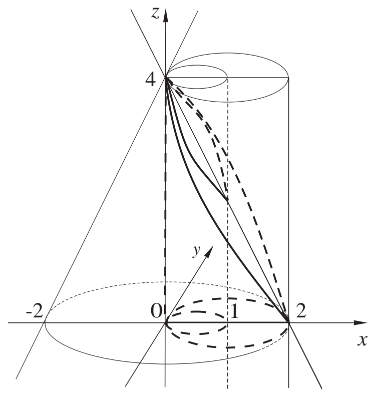
Kako je Tejlorov red $f(x) = \sum\limits_{k=0}^{\infty} \frac{f^{(k)}(z)}{k!}(x-z)^k$, funkcije $f(x)$ u tački $x = z$ jedinstven, imamo da je

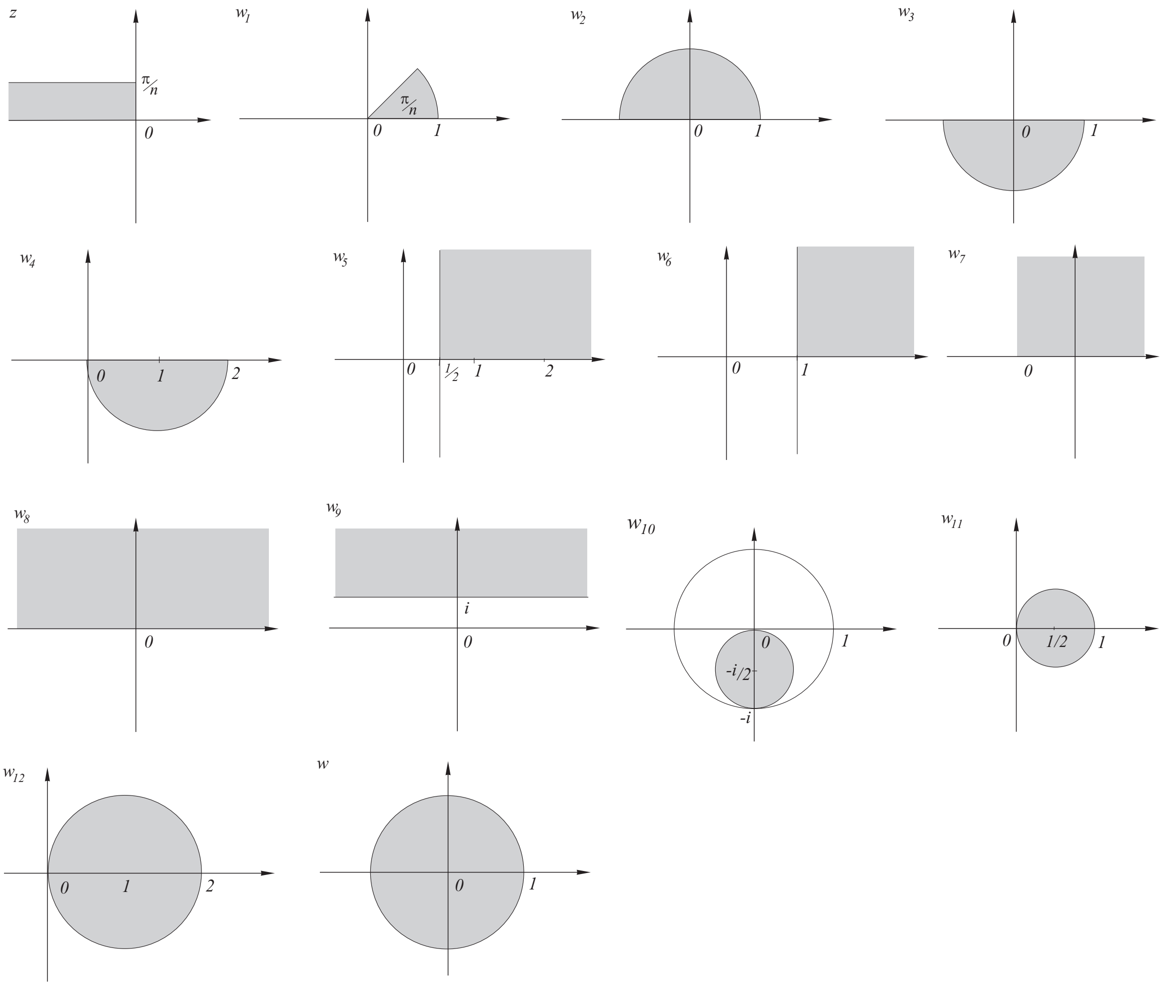
$$\sum_{n=[\frac{k}{2}]}^{\infty} \binom{2n+1}{k} \frac{(2n-1)!!}{(2n+1)(2n)!!} z^{2n+1-k} = \frac{f^{(k)}(z)}{k!}, \quad k \in \mathbb{N}_0,$$

pa je za $k = 2$

$$\sum_{n=1}^{\infty} \binom{2n+1}{2} \frac{(2n-1)!!}{(2n+1)(2n)!!} z^{2n-1} = \frac{(\arcsin z)''}{2!} = \frac{z}{2(1-z^2)\sqrt{1-z^2}}.$$

2. $V = \iiint\limits_V dxdydz = \int\limits_{-\pi/2}^{\pi/2} d\varphi \int\limits_{2\cos\varphi}^{4\cos\varphi} \rho d\rho \int\limits_0^{4-\rho} dz$
$= \int\limits_{\pi/2}^{\pi/2} d\varphi \int\limits_{2\cos\varphi}^{4\cos\varphi} (4-\rho)\rho d\rho = \int\limits_{-\pi/2}^{\pi/2} (2\rho^2 - \frac{\rho^3}{3}\Big|_{2\cos\varphi}^{4\cos\varphi}) d\varphi$
$= \int\limits_{-\pi/2}^{\pi/2} (24\cos^2\varphi - \frac{56}{3}\cos^3\varphi) d\varphi = \int\limits_{-\pi/2}^{\pi/2} (12(1+\cos 2\varphi) -$
$\frac{56}{3}(1-\sin^2\varphi)\cos\varphi) d\varphi = 12\pi - \frac{224}{9}.$





$w = f(z) = \frac{i(1-(e^z)^n)^2-(1+(e^z)^n)^2}{i(1-(e^z)^n)^2+(1+(e^z)^n)^2} = \frac{i-(\frac{1+(e^z)^n}{1-(e^z)^n})^2}{i+(\frac{1+(e^z)^n}{1-(e^z)^n})^2} = \frac{2i}{i+(\frac{2}{1-(e^z)^n}-1)^2} - 1.$

$w_1 = e^z = e^x(\cos y + i\sin y)$, $w_2 = z^n$, $w_3 = -w_1$, $w_4 = 1 + w_2$, $w_5 = \frac{1}{w_4}$, $w_6 = 2w_5$, $w_7 = w_6 - 1$, $w_8 = (w_7)^2$, $w_9 = i + w_8$, $w_{10} = \frac{1}{w_9}$, $w_{11} = iw_{10} = e^{i\frac{\pi}{2}}w_{10}$, $w_{12} = 2w_{11}$, $w = w_{12} - 1$.

$f(G) = \{w \in \mathbb{C} : |w| < 1\}.$