MATHEMATICS II -May 2007 Čanj

1. Find the region of convergence and the sum of the following series

$$\sum_{n=1}^{\infty} \frac{(3+(-1)^n)^n}{n} x^n.$$

2. If S is the external side of the surface $|x-y+z|+|y-z+x|+|z-x+y|=1$, find

$$I = \iint_S (x-y+z)dydz + (y-z+x)dzdx + (z-x+y)dxdy.$$

3. Determine $f(G)$, if $f(z) = \frac{2}{\pi} \ln \dfrac{z-3}{z+3}$ and $G = \{z \in \mathbb{C} : |z| < 3, \arg z \in (0, \pi)\}$.

MATEMATIKA II -maj 2007 Čanj

1. Naći oblast konvergencije i sumu reda

$$\sum_{n=1}^{\infty} \frac{(3+(-1)^n)^n}{n} x^n.$$

2. Ako je S spoljašnja strana površi $|x-y+z|+|y-z+x|+|z-x+y|=1$, izračunati

$$I = \iint_S (x-y+z)dydz + (y-z+x)dzdx + (z-x+y)dxdy.$$

3. Odrediti $f(G)$, ako je $f(z) = \frac{2}{\pi} \ln \dfrac{z-3}{z+3}$ i $G = \{z \in \mathbb{C} : |z| < 3, \arg z \in (0, \pi)\}$.

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Rešenje:

1. $\frac{1}{R} = \limsup \sqrt[n]{|a_n|} = \limsup \frac{3+(-1)^n}{\sqrt[n]{n}} = \max\{\lim\limits_{k\to\infty} \frac{4}{\sqrt[2k]{2k}}, \lim\limits_{k\to\infty} \frac{2}{\sqrt[2k-1]{2k-1}}\} = 4 \Rightarrow R = \frac{1}{4}$. Interval konvergencije je $(-\frac{1}{4}, \frac{1}{4})$.

Za $x = \frac{1}{4}$, red se svodi na: $\sum\limits_{n=1}^{\infty} \alpha_n$, $\alpha_n = \frac{1}{n}\Big(\frac{3+(-1)^n}{4}\Big)^n$.

$S_{2n} = \sum\limits_{k=1}^{2n} \frac{1}{k}\Big(\frac{3+(-1)^k}{4}\Big)^k = \alpha_1 + \ldots + \alpha_{2n-1} + \alpha_2 + \ldots + \alpha_{2n} \geq \alpha_2 + \ldots + \alpha_{2k} = \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2n} = \frac{1}{2}\sum\limits_{k=1}^{n} \frac{1}{k}$.

Kako $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{1}{k} = \infty \Rightarrow \lim\limits_{n\to\infty} S_{2n} = +\infty$, niz parcijalnih suma reda $\sum \alpha_n$, $(\alpha_n > 0)$ divergira (njegov podniz divergira), pa i on divergira.

Za $x = -\frac{1}{4}$, red se svodi na: $\sum\limits_{n=1}^{\infty} \beta_n$, $\beta_n = \frac{1}{n}\Big(-\frac{3+(-1)^n}{4}\Big)^n$.

$S_{2n} = \sum\limits_{k=1}^{2n} \frac{1}{k}\Big(-\frac{3+(-1)^k}{4}\Big)^k = \beta_1 + \ldots + \beta_{2n-1} + \beta_2 + \ldots + \beta_{2n} = -\sum\limits_{l=1}^{n} \frac{1}{2l-1}\frac{1}{2^{2l-1}} + \sum\limits_{l=1}^{n} \frac{1}{2l}$.

Red $\frac{1}{2}\sum\limits_{l=1}^{\infty} \frac{1}{l}$ divergira ∞, a red $\sum\limits_{l=1}^{\infty} \frac{1}{2l-1}\frac{1}{2^{2l-1}}$ konvergira, pa je $\lim\limits_{n\to\infty} S_{2n} = \infty$, tj. $\sum \beta_n$ divergira.

Oblast konvergencije je $(-\frac{1}{4}, \frac{1}{4})$

$\sum\limits_{n=1}^{\infty} \frac{(3+(-1)^n)^n}{n} x^n = \sum\limits_{n=1}^{\infty} (3+(-1)^n)^n \int\limits_0^x t^{n-1} dt = \int\limits_0^x \sum\limits_{n=1}^{\infty} \frac{1}{t} [(3+(-1)^n)t]^n dt = \int\limits_0^x \frac{1}{t} \sum\limits_{n=1}^{\infty} [(3+(-1)^n)t]^n dt = \int\limits_0^x \frac{1}{t} \Big(\sum\limits_{m=1}^{\infty} (4t)^{2m} + \sum\limits_{m=1}^{\infty} (2t)^{2m-1} \Big) dt = \int\limits_0^x \frac{1}{t} \Big(\frac{16t^2}{1-16t^2} + \frac{2t}{1-4t^2} \Big) dt = -\frac{1}{2} \int\limits_0^x \frac{-32t}{1-16t^2} dt + \int\limits_0^x \Big(\frac{1}{1-2t} + \frac{1}{1+2t} \Big) dt = -\frac{1}{2} \ln(1-16x^2) + \frac{1}{2} \ln \frac{1+2x}{1-2x} = S(x)$, $x \in (-\frac{1}{4}, \frac{1}{4}) \setminus \{0\}$.

Za $x = 0$ suma reda je 0, pa je $\lim\limits_{x \to \infty} S(x) = 0$, tj. $S(x)$ je neprekidna funkcija, pa je suma reda $S(x)$, $x \in (-\frac{1}{4}, \frac{1}{4})$:

$$S(x) = \frac{1}{2} \ln \frac{1+2x}{(1-2x)(1-16x^2)}.$$

2. Oblast $V = \{(x,y,z) : |x-y+z| + |y-z+x| + |z-x+y| \leq 1\}$ je zatvorena i ograničena sa deo po deo glatkom zadatom površi S (koja se ne samopreseca), funkcije $P = x-y+z$, $Q = y-z+x$, $R = z-x+y$ definisane i neprekidne nad V imaju neprekidne prve izvode. Sa obzirom da je S spoljašnja strana površi, na osnovu teoreme Gaus-Ostrogradski imamo

$$I = \iiint\limits_V (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}) dxdydz = 3 \iiint\limits_V dxdydz.$$

Posmatrajmo transformaciju $F((x,y,z)) = (u,v,w)$, definisanu sa $u = x-y+z$, $v = y-z+x$, $w = z-x+y$. Tako imamo $F(V) = V' = \{(u,v,w) : |u| + |v| + |w| \leq 1\}$ i

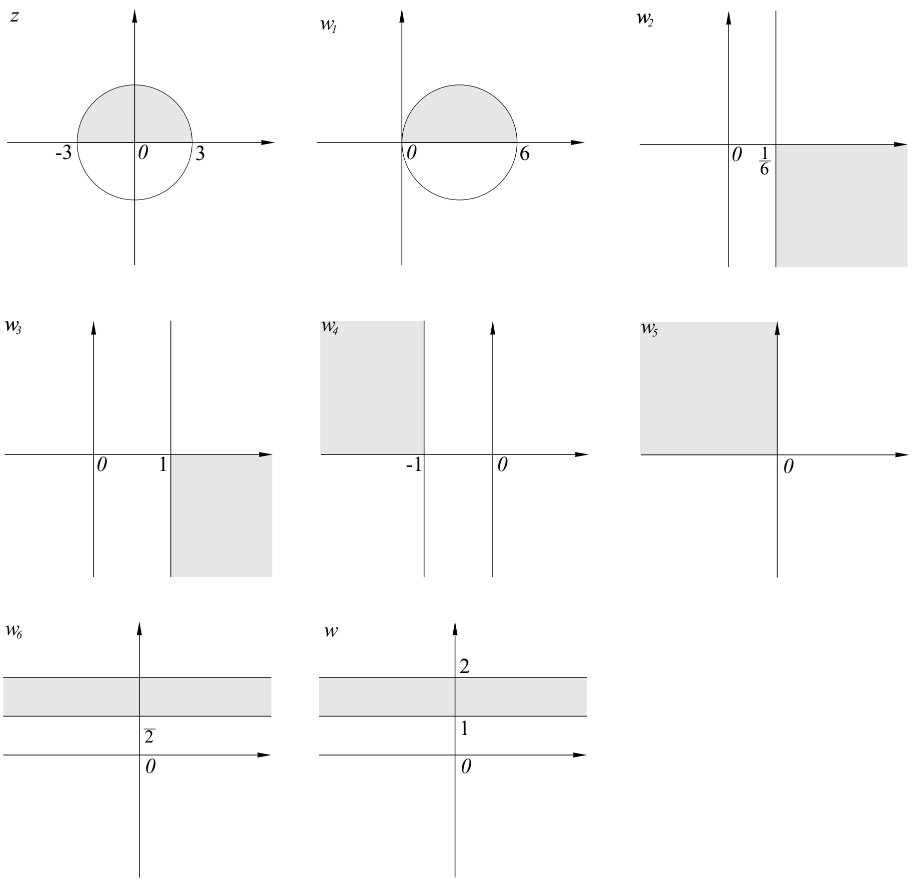
$$J = \frac{D(x,y,z)}{D(u,v,w)} = \frac{1}{\frac{D(u,v,w)}{D(x,y,z)}} = \frac{1}{\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix}} = \frac{1}{4}.$$

Ako je $V'' = \{(u,v,w) \in (\mathbb{R}_0^+)^3 : u+v+w \leq 1\}$, tada je

$$I = \frac{3}{4} \iiint\limits_{V'} dudvdw = 6 \iiint\limits_{V''} dudvdw = 6 \int\limits_0^1 du \int\limits_0^{1-u} dv \int\limits_0^{1-u-v} dw = 6 \int\limits_0^1 du \int\limits_0^{1-u} (1-u-v) dv$$

$$= 6 \int\limits_0^1 (v(1-u) - \frac{v^2}{2}) \Big|_0^{1-u} du = 6 \int\limits_0^1 \frac{1}{2} (1-u)^2 du = \frac{6}{2} \cdot \frac{(u-1)^3}{3} \Big|_0^1 = 0 - (-1)^3 = 1.$$

3. $w = \frac{2}{\pi} \ln \frac{z-3}{z+3} = \frac{2}{\pi} \ln(1 - \frac{6}{z+3})$.
$w_1 = z + 3$.
$w_2 = \frac{1}{w_1}$.
$w_3 = 6w_2$.
$w_4 = -w_3$.
$w_5 = 1 + w_4$.
$w_6 = \ln w_5 = \ln |w_5| + i \arg w_5$; $y = 0, x < 0 \longrightarrow \ln |x| + i\pi = u + iv$, $u \in [-\infty, \infty]$, $v = \pi$; $x = 0, y \geq 0 \longrightarrow \ln |y| + i\frac{\pi}{2} = u + iv$, $u \in [-\infty, \infty]$, $v = \frac{\pi}{2}$; $-1 + i \longrightarrow \ln \sqrt{2} + \frac{3\pi}{4} i$.
$w = \frac{2}{\pi} w_6$.



Oblast $\{z \in \mathbb{C} : |z| < 3, \arg z \in (0, \pi)\}$ funkcijom $w = f(z)$ preslikava se u oblast $\{w \in \mathbb{C} : 1 < \operatorname{Im} w < 2\}$.

Oblast $G = \{z : |z| < 3, \operatorname{Im} z > 0\}$, možemo preslikati preslikavanjem $w_5 = \frac{z-3}{z+3}$ koristeći da je $z = -3\frac{w_5+1}{w_5-1}$ i: $|z| < 3 \Leftrightarrow 3\frac{|w_5+1|}{|w_5-1|} < 3 \Leftrightarrow |w_5 + 1| < |w_5 - 1| \Leftrightarrow |w_5 + 1|^2 < |w_5 - 1|^2 \Leftrightarrow (w_5 + 1)(\overline{w_5} + 1) < (w_5 - 1)(\overline{w_5} - 1) \Leftrightarrow w\overline{w_5} + \overline{w_5} + w_5 + 1 < w_5\overline{w_5} - \overline{w_5} - w_5 + 1 \Leftrightarrow 2(w_5 + \overline{w_5}) < 0 \Leftrightarrow \operatorname{Re} w_5 < 0$.

$0 < \operatorname{Im} z = \frac{1}{2i}(z - \overline{z}) = \frac{1}{2i}(-3\frac{w_5+1}{w_5-1} - \overline{(-3\frac{w_5+1}{w_5-1})}) = \frac{3}{2i}(\frac{\overline{w_5}+1}{\overline{w_5}-1} - \frac{w_5+1}{w_5-1}) = \frac{w_5-\overline{w_5}}{2i} \cdot \frac{6}{(w_5-1))\overline{(w_5-1)}} = \frac{6}{|w_5-1|^2} \cdot \operatorname{Im} w_5$ $\Leftrightarrow \operatorname{Im} w_5 > 0$.