MATHEMATICS II -May 2005 Kopaonik-Konaci

1. If $a, d \in \mathbb{Q}^+$, find the interval of convergence and the sum of the series

$$\sum_{n=1}^{\infty} \frac{a(a+d)...(a+(n-1)d)}{d \cdot 2d \cdot ... \cdot nd} x^n,$$

2. Find volume of the following region

$$V = \{(x,y,z) \in \mathbb{R}^3 : \ \big((\frac{x}{a})^{\frac{2}{3}} + (\frac{y}{b})^{\frac{2}{3}}\big)^3 + (\frac{z}{c})^2 \leq 1\} \quad (a,b,c>0).$$

3. If $n \in \mathbb{N}$, calculate the given integral $I = \int\limits_0^{\pi/2} \cos^{2n} x dx$, using complex analysis methods.

MATEMATIKA II -maj 2005 Kopaonik-Konaci

1. Ako je $a, d \in \mathbb{Q}^+$, naći interval konvergencije i sumu reda

$$\sum_{n=1}^{\infty} \frac{a(a+d)...(a+(n-1)d)}{d \cdot 2d \cdot ... \cdot nd} x^n.$$

2. Naći zapreminu oblasti

$$V = \{(x,y,z) \in \mathbb{R}^3 : \ \big((\frac{x}{a})^{\frac{2}{3}} + (\frac{y}{b})^{\frac{2}{3}}\big)^3 + (\frac{z}{c})^2 \leq 1\} \quad (a,b,c>0).$$

3. Izračunati integral $I = \int\limits_0^{\pi/2} \cos^{2n} x dx, \quad n \in \mathbb{N}$, koristeći metode kompleksne analize.

Mentor takmičenja: Nebojša M. Ralević

Rešenje:

1. Poluprečnik konvergencije reda $\sum\limits_{n=1}^{\infty} a_n x^n$, $a_n = \frac{a(a+d)...(a+(n-1)d)}{d \cdot 2d \cdot ... \cdot nd}$ je jednak

$$\rho = \lim_{n\to\infty} |\frac{a_n}{a_{n+1}}| = \lim_{n\to\infty} \frac{\frac{a(a+d)...(a+(n-1)d)}{d \cdot 2d \cdot ... \cdot nd}}{\frac{a(a+d)...(a+(n-1)d)(a+nd)}{d \cdot 2d \cdot ... \cdot nd \cdot (n+1)d}} = \lim_{n\to\infty} \frac{(n+1)d}{a+nd} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})d}{\frac{a}{n}+d} = 1,$$

pa red konvergira za svako x iz intervala konvergencije $(-\rho, \rho) = (-1,1)$.

Kako je

$$a_n = \frac{1}{n!}\frac{a}{d}(\frac{a}{d}+1)...(\frac{a}{d}+n-1) = \frac{(-1)^n}{n!}(-\frac{a}{d})(-\frac{a}{d}-1)...(-\frac{a}{d}-n+1) = (-1)^n \binom{-a/d}{n},$$

jer je $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)...(\alpha-n+1}{n!}$, to je suma reda

$$S(x) = \sum_{n=1}^{\infty} (-1)^n \binom{-a/d}{n} x^n = \sum_{n=0}^{\infty} \binom{-a/d}{n} (-x)^n - 1 = (1-x)^{-\frac{a}{d}} - 1, \ \ x \in (-1,1).$$

2. Transformacija $x = ar\cos^3\varphi \sin\theta$, $y = br\sin^3\varphi\sin\theta$, $z = cr\cos\theta$ preslikava oblast $G = \{(r,\varphi,\theta) \in [0,\infty) \times [0,2\pi] \times [0,\pi] : \ r^2 \leq 1\} = \{(r,\varphi,\theta) \in [0,1] \times [0,2\pi] \times [0,\pi]\}$ na oblast V i za datu transformaciju je

$J(r,\varphi,\theta)=\frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)}=\begin{vmatrix} a\cos^3\varphi\sin\theta & -3ar^3\cos^2\varphi\sin\varphi\sin\theta & ar\cos^3\varphi\cos\theta \\ b\sin^3\varphi\sin\theta & 3br\sin^2\varphi\cos\varphi\sin\theta & br\sin^2\theta\cos\theta \\ c\cos\theta & 0 & -cr\sin\theta \end{vmatrix}$

$=3abcr^2\sin^2\varphi\cos^2\varphi\begin{vmatrix} \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \\ -\sin\varphi\sin\theta & \cos\varphi\sin\theta & 0 \\ \cos\varphi\cos\theta & \sin\varphi\cos\theta & -\sin\theta \end{vmatrix}=-3abcr^2\sin^2\varphi\cos^2\varphi\sin\theta.$

Dakle, $|J|=3abcr^2\sin^2\varphi\sin\theta\cos^2\varphi$. Sada je
$\Delta V=3abc\iiint\limits_G dxdydz\, r^2\sin^2\varphi\sin\theta\cos^2\varphi drd\varphi d\theta \quad =3abc\int_0^{2\pi}\sin^2\varphi\cos^2\varphi d\varphi\int_0^{\pi}\sin\theta d\theta\int_0^1 r^2dr=\frac{abc\pi}{2}$,
jer je $\int\sin^2\varphi\cos^2\varphi d\varphi=\frac{1}{4}\int\sin^2 2\varphi d\varphi=\frac{1}{4}\int\frac{1-\cos 4\varphi}{2}d\varphi=\frac{1}{8}(\varphi-\frac{1}{4}\sin 4\varphi)$.

3. Primetimo najpre da je $\int\limits_0^{\pi/2}\cos^{2n}xdx=\frac{1}{4}\int\limits_0^{2\pi}\cos^{2n}xdx=\frac{1}{4}J$. Za $z=e^{ix}$, je $dz=ie^{ix}dx$, tj. $dx=\frac{1}{iz}dz$ i $\cos x=\frac{z^2+1}{2z}$. Tada je

$$J=\int\limits_c\Big(\frac{z^2+1}{2z}\Big)^{2n}\frac{1}{iz}dz=-\frac{i}{2^{2n}}\int\limits_c\frac{(z^2+1)^{2n}}{z^{2n+1}}dz,$$

gde je c pozitivno orijentisana kružnica $\{z\in\mathbb{C}:\ |z|=1\}$.

Tačka $z=0$ je pol reda $2n+1$ funkcije

$$f(z)=\frac{(z^2+1)^{2n}}{z^{2n+1}}=\frac{1}{z^{2n+1}}\sum_{k=0}^{2n}\binom{2n}{k}z^{2k}=\sum_{k=0}^{2n}\binom{2n}{k}z^{2k-2n-1},$$

pa je prethodna suma zapravo razvoj funkcije $f(z)$ u Loranov red u okolini tačke $z=0$. Kako je za $2k-2n-1=-1$, tj. $k=n$ koeficijent uz z^{-1} jednak $a_{-1}=\binom{2n}{n}$, to je

$J=-\frac{i}{2^{2n}}\cdot 2\pi i\operatorname*{Res}_{z=0}f(z)=\frac{\pi}{2^{2n-1}}\binom{2n}{n}=\frac{\pi}{2^{2n-1}}\frac{(2n)!}{n!n!}=\frac{\pi}{2^{-1}}\frac{(2n)!!(2n-1)!!}{2^n n!2^n n!}=2\pi\frac{(2n)!!(2n-1)!!}{((2n)!!)^2}=2\pi\frac{(2n-1)!!}{(2n)!!}$.

Na kraju je $I=\int\limits_0^{\pi/2}\cos^{2n}xdx=\frac{\pi}{2}\frac{(2n-1)!!}{(2n)!!}$.