MATHEMATICS I -May 2010 Čanj

1. Examine the existence of a cube such that two neighboring vertices belong to the line $p: \frac{x+12}{8} = \frac{y+1}{1} = \frac{z}{4}$, and two vertices of the cube belong to the line $q: \frac{x-19}{11} = \frac{y+14}{-11} = \frac{z-2}{1}$? If that cube exists, find the coordinates of all vertices.

2. Examine detaily the function
$$f(x) = \operatorname{arcctg} \frac{x^2}{x^2-1},$$
and draw its graphic.

3. Find the integral:
$$\int \frac{x}{(1-x^3)\sqrt{1-x^2}} dx.$$

MATEMATIKA I -maj 2010 Čanj

1. Da li postoji kocka čija dva susedna temena pripadaju pravoj $p: \frac{x+12}{8} = \frac{y+1}{1} = \frac{z}{4}$, a dva temena te iste kocke pripadaju pravoj $q: \frac{x-19}{11} = \frac{y+14}{-11} = \frac{z-2}{1}$? Ako postoji naći koordinate njenih temena.

2. Detaljno ispitati funkciju
$$f(x) = \operatorname{arcctg} \frac{x^2}{x^2-1},$$
i nacrtati njen grafik.

3. Naći integral:
$$\int \frac{x}{(1-x^3)\sqrt{1-x^2}} dx.$$

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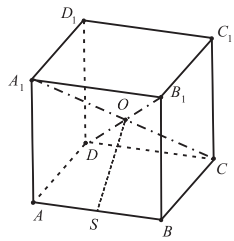
Rešenje:

1. Ako označimo sa O centar kocke i S sredinu ivice $AB \subset p$, kao na slici, možemo zaključiti da prave p i q mogu biti paralelne, ortogonalne, ili da je ugao izmedju njih $\frac{\pi}{4}$, odnosno $\arccos\frac{\sqrt{3}}{3}$ (ugao glavne dijagonale i ivice). Iz jednačina pravih p i q sledi da su njihov vektori pravaca i pripadne tačke $\overrightarrow{p} = (8, 1, 4)$, $\overrightarrow{q} = (11, -11, 1)$, $P(-12, -1, 0)$, $Q(19, -14, 2)$. Sada imamo da je $\overrightarrow{PQ} = (31, -13, 2)$, $|\overrightarrow{p}| = 9$, $\overrightarrow{q} = 9\sqrt{3}$,



$$\overrightarrow{p} \times \overrightarrow{q} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 8 & 1 & 4 \\ 11 & -11 & 1 \end{vmatrix} = (45, 36, -99) = 9(5, 4, -1),$$

$(\overrightarrow{p} \times \overrightarrow{q}) \cdot \overrightarrow{PQ} = 9(5, 4, -1) \cdot (31, -13, 2) = 9(155 - 52 - 22) = 729 \neq 0 \;\Rightarrow\; p, q$ su mimoilazne prave za koje je

$$\angle(\overrightarrow{p}, \overrightarrow{q}) = \arccos\frac{\overrightarrow{p} \cdot \overrightarrow{q}}{|\overrightarrow{p}||\overrightarrow{q}|} = \arccos\frac{8 \cdot 11 + 1 \cdot (-11) + 4 \cdot 1}{9 \cdot 9\sqrt{3}} = \arccos\frac{\sqrt{3}}{3}.$$

Kako su prave p i q mimoilazne, to se na pravoj q nalaze temena A_1 i C ili B_1 i D, odnosno centar O pripada q. Prava SO je zajedni;ka normala, pa je

$$d(S, O) = d(p, q) = \frac{(\overrightarrow{p} \times \overrightarrow{q}) \cdot \overrightarrow{PQ}}{|\overrightarrow{p} \times \overrightarrow{q}|} = \frac{729}{9 \cdot 9\sqrt{2}} = \frac{9}{\sqrt{2}},$$

te kako je $d(S, O)$ polovina bočne dijagonale, imamo da je $\frac{1}{2}a\sqrt{2} = \frac{9}{\sqrt{2}}$, tj. $a = 9$.

$\{O\} = q \cap \sigma$, $\sigma = r(p, OS)$

$$\vec{n}_\sigma \parallel \overrightarrow{p} \times (\overrightarrow{p} \times \overrightarrow{q}) \parallel \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 8 & 1 & 4 \\ 5 & 4 & -11 \end{vmatrix} = (-27, 108, 27) \parallel (1, -4, -1),$$

$\sigma:\ (x - (-12)) - 4(y - (-1)) - (z - 0) = 0$, tj. $\sigma:\ x - 4y - z + 8 = 0$, $q:\ x = 11t + 19,\ y = -11t - 14,\ z = t + 2$, $\sigma \cap q:\ 11t + 19 + 44t + 56 - t - 2 + 8 = 0$, $t = -\frac{3}{2}$, $O(\frac{5}{2}, \frac{5}{2}, \frac{1}{2})$.

$\{S\} = p \cap \pi$, $\pi = r(q, OS)$

$$\vec{n}_\pi \parallel \overrightarrow{q} \times (\overrightarrow{p} \times \overrightarrow{q}) \parallel \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 11 & -11 & 1 \\ 5 & 4 & -11 \end{vmatrix} = (117, 126, 99) \parallel (13, 14, 11),$$

$\pi:\ 13(x - 19)) + 14(y - (-14)) + 11(z - 2) = 0$, tj. $\pi:\ 13x + 14y + 11z - 73 = 0$, $p:\ x = 8t - 12,\ y = t - 1,\ z = 4t$, $\pi \cap p:\ 104t - 156 + 14t - 14 + 44t - 73 = 0$, $t = \frac{3}{2}$, $S(0, \frac{1}{2}, 6)$.

Sada imamo da je je

$$\overrightarrow{r}_{A,B} = \overrightarrow{r}_S \pm \frac{a}{2}\frac{1}{p}\overrightarrow{p} = (0, \frac{1}{2}, 6) \pm \frac{9}{2} \cdot \frac{1}{9}(8, 1, 4) \;\; A(4, 1, 8),\ B(-4, 0, 4).$$

$$\overrightarrow{r}_O = \frac{1}{2}(\overrightarrow{r}_A + \overrightarrow{r}_{C_1}) \;\Rightarrow\; \overrightarrow{r}_{C_1} = 2\overrightarrow{r}_O - \overrightarrow{r}_A = (1, 4, -7),\;\; C_1(1, 4, -7).$$

$$\overrightarrow{r}_{D_1} = 2\overrightarrow{r}_O - \overrightarrow{r}_B = (9, 5, 8),\;\; B(9, 5, 8).$$

$$\overrightarrow{r}_{A_1,C} = \overrightarrow{r}_O \pm \frac{a\sqrt{3}}{2}\frac{1}{q}\overrightarrow{q} = (\frac{5}{2}, \frac{5}{2}, \frac{1}{2}) \pm \frac{9\sqrt{3}}{2} \cdot \frac{1}{9\sqrt{3}}(11, -11, 1) \;\; A_1(8, -3, 1),\ C(-3, 8, 0).$$

$$\overrightarrow{r}_B - \overrightarrow{r}_A = \overrightarrow{r}_C - \overrightarrow{r}_D \;\Rightarrow\; \overrightarrow{r}_D = \overrightarrow{r}_C - \overrightarrow{r}_B + \overrightarrow{r}_A = (5, 9, 4),\;\; D(5, 9, 4).$$

$$\overrightarrow{r}_{B_1} = 2\overrightarrow{r}_O - \overrightarrow{r}_D = (0, -4, -3),\;\; B_1(0, -4, -3).$$

2. Funkcija arcctg je uvek definisana, ali $\frac{x^2}{x^2-1}$ nije za $x^2 - 1 = 0$, tj. domen funkcije f je $\mathcal{D}_f = \mathbb{R} \setminus \{-1, 1\}$. Zbog $f(-x) = f(x)$ je funkcija f parna, odnosno grafik joj je simetričan u odnosu na $y-$osu, te je dovoljno posmatrati funkciju za $x \geq 0$.

Funkcija je pozitivna za sve $x \in \mathcal{D}_f$.

Na osnovu $\lim\limits_{x\to+\infty} \operatorname{arcctg} x = 0$, $\lim\limits_{x\to-\infty} \operatorname{arcctg} x = \pi$ sledi

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} \operatorname{arcctg}\frac{x^2}{x^2 - 1} = 0, \quad \lim_{x\to1^-} f(x) = \lim_{x\to1^-} \operatorname{arcctg}\frac{x^2}{x^2 - 1} = \pi,$$

pa u tački $x = 1$ funkcija ima prekid prve vrste (leva i desna granična vrednost funkcije postoje, ali nisu iste), te nemamo vertikalnu asimptotu. Zbog

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \operatorname{arcctg} \frac{x^2}{x^2-1} = \operatorname{arcctg}\ 1 = \frac{\pi}{4},$$

sledi da je prava $y = \frac{\pi}{4}$ horizontalna asimptota funkcije f kada $x \to +\infty$.

$f'(x) = -\frac{1}{1+(\frac{x^2}{x^2-1})^2} \cdot \frac{2x(x^2-1)-x^2\cdot 2x}{(x^2-1)^2} = -\frac{1}{\frac{x^4-2x^2+1+x^4}{(x^2-1)^2}} \cdot \frac{-2x}{(x^2-1)^2} = \frac{2x}{2x^4-2x^2+1}.$

Kako je $2x^4 - 2x^2 + 1 = x^4 + (x^2-1)^2 > 0$, to je prvi izvod uvek definisan. Otuda $f'(x) = 0 \Leftrightarrow x = 0$, $x > 0 \Rightarrow f'(x) > 0$ (funkcija raste), te je zbog parnosti funkcije u $x = 0$ njen minimum $y_{min} = \frac{\pi}{2}$.

$$\lim_{x\to 1^\pm} f'(x) = \lim_{x\to 1^\pm} \frac{2x}{2x^4-2x^2+1} = 2,$$

pa tangenta na krivu i levo i desno od tačke $(0, \frac{\pi}{2})$ zaklapa sa pozitivnim delom x-ose ugao čiji je kotangens 2.

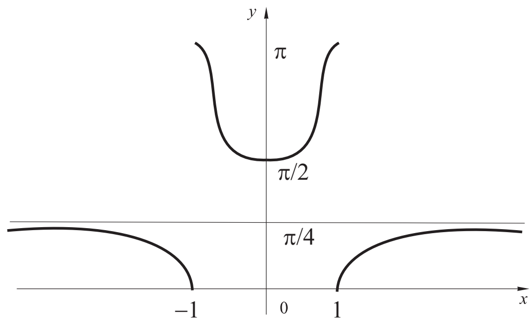
$f''(x) = \frac{2(2x^4-2x^2+1)-2x(8x^3-4x)}{(2x^4-2x^2+1)^2} = \frac{-12x^4+4x^2+2}{(2x^4-2x^2+1)^2} = \frac{2(1+2x^2-6x^4)}{(2x^4-2x^2+1)^2}.$

$$f''(x) = 0 \Leftrightarrow 6(x^2)^2 - 2x^2 - 1 = 0$$

$$\Leftrightarrow x^2 = \frac{1-\sqrt{7}}{6} < 0 \vee x^2 = \frac{1+\sqrt{7}}{6} > 0 \Leftrightarrow x = \sqrt{\frac{1+\sqrt{7}}{6}} > 0.$$

Tačka $(\sqrt{\frac{1+\sqrt{7}}{6}}, \operatorname{arcctg}(-\frac{2+\sqrt{7}}{3}))$ je prevojna tačka funkcije.

Za $x \in (0, \sqrt{\frac{1+\sqrt{7}}{6}})$ funkcija je konveksna.



3. Primitivnu funkciju funkcije $f(x)$ tražimo nad intervalom $I \subset (-1, 1)$.

$$\frac{x}{1-x^3} = \frac{A}{1-x} + \frac{Bx+C}{1+x+x^2} = \frac{(A-B)x^2 + (A+B-C)x + (A+C)}{(1-x)(1+x+x^2)}$$

$$A - B = 0, A + B - C = 1, A + C = 0 \Leftrightarrow A = B = \frac{1}{3}, C = -\frac{1}{3},$$

$$\int \frac{x}{(1-x^3)\sqrt{1-x^2}}dx = \frac{1}{3}[\int \frac{1}{(1-x)\sqrt{1-x^2}}dx + \int \frac{x-1}{(1+x+x^2)\sqrt{1-x^2}}dx] = \frac{1}{3}(I_1 + I_2).$$

$t = \arcsin x,\ x = \sin t,\ x \in (-1, 1),\ t \in (-\frac{\pi}{2}, \frac{\pi}{2}),\ dt = \frac{dx}{\sqrt{1-x^2}}$

$$I_1 = \int \frac{dt}{1-\sin t} = \int \frac{1+\sin t}{1-\sin^2 t}dt = \int \frac{1}{\cos^2 t}dt + \int \frac{\sin t}{\cos^2 t}dt$$

$$= \operatorname{tg} t - \int \frac{du}{u^2} = \operatorname{tg} t + \frac{1}{\cos t} + C = \frac{\sin t + 1}{\sqrt{1-\sin^2 t}} + C = \frac{x+1}{\sqrt{1-x^2}} + C\sqrt{\frac{1+x}{1-x}} + C.$$

$$I_2 = -\int \frac{1-x}{(1+x+x^2)\sqrt{1-x^2}}dx = -\int \frac{\sqrt{1-x^2}}{(1+x+x^2)\sqrt{1-x^2}}dx = -\int \frac{1}{1+x+x^2}\sqrt{\frac{1-x}{1+x}}dx$$

$t = \sqrt{\frac{1-x}{1+x}},\ x = \frac{1-t^2}{1+t^2},\ dx = \frac{-4t}{(1+t^2)^2}dt$

$$I_2 = -\int \frac{1}{1+\frac{1-t^2}{1+t^2}+\frac{(1-t^2)^2}{(1+t^2)^2}) \cdot t \cdot \sqrt{1-x^2}} \frac{-4t}{(1+t^2)^2}dt = \int \frac{4t^2}{2(1+t^2)+(1-t^2)^2}dt = \int \frac{4t^2}{t^4+3}dt = \int \frac{4\sqrt{3}s^2}{3s^4+3}\sqrt[4]{3}ds$$

$$= \frac{4}{\sqrt[4]{3}}\int \frac{s^2}{s^4+1}ds. = \frac{4}{\sqrt[4]{3}}\int \frac{1}{2\sqrt{2}}(\frac{s}{s^2-\sqrt{2}s+1} - \frac{s}{s^2+\sqrt{2}s+1})ds = \frac{\sqrt{2}}{2\sqrt[4]{3}}(\int \frac{2s-\sqrt{2}+\sqrt{2}}{s^2-\sqrt{2}s+1}ds - \int \frac{2s+\sqrt{2}-\sqrt{2}}{s^2+\sqrt{2}s+1}ds)$$

$$= \frac{\sqrt{2}}{2\sqrt[4]{3}}[\ln|s^2-\sqrt{2}s+1| + \sqrt{2}\int \frac{ds}{(s-\sqrt{2}/2)^2+(\sqrt{2}/2)^2}ds - \ln|s^2+\sqrt{2}s+1| + \sqrt{2}\int \frac{ds}{(s+\sqrt{2}/2)^2+(\sqrt{2}/2)^2}ds]$$

$$= \frac{\sqrt{2}}{2\sqrt[4]{3}}[\ln\frac{s^2-\sqrt{2}s+1}{s^2+\sqrt{2}s+1} + 2\operatorname{arctg}\frac{s-\sqrt{2}/2}{\sqrt{2}/2} + 2\operatorname{arctg}\frac{s+\sqrt{2}/2}{\sqrt{2}/2}] + C = \frac{\sqrt{2}}{2\sqrt[4]{3}}[\ln\frac{s^2-\sqrt{2}s+1}{s^2+\sqrt{2}s+1} + 2\operatorname{arctg}\frac{\sqrt{2}s}{1-s^2}] + C.$$