MATHEMATICS I -May 2005 Kopaonik-Konaci

1. Find the vertices A, B and C of a regular polyhedron $ABCD$, whose the center of gravity $T(\frac{1}{2}, \frac{3}{2}, \frac{3}{2})$, the vertex $D(0,2,2)$, and the vertex A the closest to the plane $\alpha : x = 0$.

2. Examine detaily the function
$$f(x) = \ln(\arccos \frac{2x}{1+x^2}),$$
and draw its graphic.

3. Find primitive function of:
$$f(x) = \frac{x}{(x^2-3x+2)\sqrt{x^2-4x+3}}.$$

MATEMATIKA I -maj 2005 Kopaonik-Konaci

1. Nađi temena A, B, i C pravilnog tetraedra $ABCD$, čije je težište $T(\frac{1}{2}, \frac{3}{2}, \frac{3}{2})$, teme $D(0,2,2)$, a teme A najbliže ravni $\alpha : x = 0$.

2. Detaljno ispitati funkciju
$$f(x) = \ln(\arccos \frac{2x}{1+x^2})$$
i nacrtati njen grafik.

3. Naći primitivnu funkciju od:
$$f(x) = \frac{x}{(x^2-3x+2)\sqrt{x^2-4x+3}}.$$

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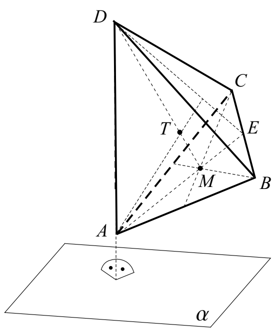
Rešenje:

1. Neka je a dužina ivice tetraedra, a M težište trougla ABC i E sredina ivice BC. Iz jednakokrakog trougla DME je visina tetraedra



$$H = DM = \sqrt{(\frac{a\sqrt{3}}{2})^2 - (\frac{1}{3}\frac{a\sqrt{3}}{2})^2} = \frac{a\sqrt{6}}{3}.$$

Iz trougla TMA je $(H-r)^2 + (\frac{2}{3}\frac{a\sqrt{3}}{2})^2 = r^2$ odakle je poluprečnik opisane sfere tetraedra $r = \frac{a\sqrt{6}}{4}$, tj. težište deli visinu u odnosu $DT : TM = 3 : 1$.
Sada imamo da je je
$$\vec{r}_M = \vec{r}_T + \frac{1}{3}\overrightarrow{DT} = (\frac{2}{3}, \frac{4}{3}, \frac{4}{3}).$$

Neka je l prava takva da $M \in l$ i $l \| (\vec{n}_\alpha \times \overrightarrow{DT}) \times \overrightarrow{DT} = -\frac{1}{4}(2,1,1)$ i neka je $\{F\} = l \cap \alpha$. Tada je $\vec{r}_F = (0,1,1)$ i $MA = \frac{1}{3}\sqrt{6} = MF$ što znači da je $A = F$.

Konačno,
$$\vec{r}_E = \vec{r}_M + \frac{1}{2}\overrightarrow{AM} = (1, \frac{3}{2}, \frac{3}{2}),$$
$$\vec{r}_{B,C} = \vec{r}_E \pm \frac{1}{2}BC\frac{\overrightarrow{DT} \times \overrightarrow{AM}}{|\overrightarrow{DT} \times \overrightarrow{AM}|},$$
odakle je $B(1,1,2)$ i $C(1,2,1)$.

2. Oblast definisanosti je $\mathcal{D}_f = R \setminus \{1\}$.
$y = 0 \Leftrightarrow \arccos z = 1 \Leftrightarrow z = \cos 1 \Leftrightarrow x^2 - \frac{2}{\cos 1}x + 1 = 0 \Leftrightarrow x_{1,2} = \frac{1 \pm \sin 1}{\cos 1}$. ($0 < x_1 \approx 0.29 < 1$, $x_2 \approx 3.41 > 1$)

$y > 0 \Leftrightarrow \arccos z > 1 \Leftrightarrow z \in [-1, \cos 1) \Leftrightarrow x \in (-\infty, x_1) \cup (x_2, \infty)$.

$\lim\limits_{x\to\pm\infty} y = \lim\limits_{x\to\pm\infty} \ln\arccos\frac{2x}{1+x^2} = \ln\arccos 0 = \ln\frac{\pi}{2}$, $\quad y = \ln\frac{\pi}{2} \approx 0,45$ je horizontalna asimptota.

$\lim\limits_{x\to 1} \ln\arccos\frac{2x}{1+x^2} = \ln\arccos 1 = -\infty$, $\ x = 1$ je vertikalna asimptota.

$$y' = \frac{1}{\arccos\frac{2x}{1+x^2}} \cdot \frac{-1}{\sqrt{1-(\frac{2x}{1+x^2})^2}} \cdot \frac{2(1+x^2)-2x\cdot 2x}{(1+x^2)^2} = \frac{2(x^2-1)}{|x^2-1|(1+x^2)} \cdot \frac{1}{\arccos\frac{2x}{1+x^2}} = \begin{cases} \frac{-2}{(1+x^2)\arccos\frac{2x}{1+x^2}}, & |x| < 1 \\ \frac{2}{(1+x^2)\arccos\frac{2x}{1+x^2}}, & |x| > 1 \end{cases},$$

$y' < 0 \ \Leftrightarrow x \in (-1, 1)$, $\ y' > 0 \ \Leftrightarrow x \in (-\infty, -1) \cup (1, +\infty)$.

$\lim\limits_{x\to -1+} y' = -\frac{1}{\pi}$, $\ \lim\limits_{x\to -1-} y' = \frac{1}{\pi}$. Za $x = -1$ prvi izvod nije definisan, ali on menja znak u okolini te tačke, pa je $(x_{max}, y_{max}) = (-1, \ln\pi)$. $\quad$ ($\ln\pi \approx 1.14$)

$$y'' = \begin{cases} \frac{4(x\arccos\frac{2x}{1+x^2}-1)}{(1+x^2)^2\arccos^2\frac{2x}{1+x^2}}, & |x| < 1 \\ \frac{-4(x\arccos\frac{2x}{1+x^2}+1)}{(1+x^2)^2\arccos^2\frac{2x}{1+x^2}}, & |x| > 1 \end{cases}.$$

Primetimo $z = \frac{2x}{1+x^2}$, $\ z' = \frac{2(1-x^2)}{(1+x^2)^2}$, $\quad |x| < 1 \Rightarrow z \nearrow$, $|x| > 1 \Rightarrow z \searrow$, te sledi

1° $x \in (-\infty, -1) \Rightarrow z = \frac{2x}{1+x^2} \in (-1, 0) \Rightarrow \arccos z \in (\frac{\pi}{2}, \pi)$
$\Rightarrow x\arccos z + 1 < -\frac{\pi}{2} + 1 < 0 \Rightarrow y'' > 0$,

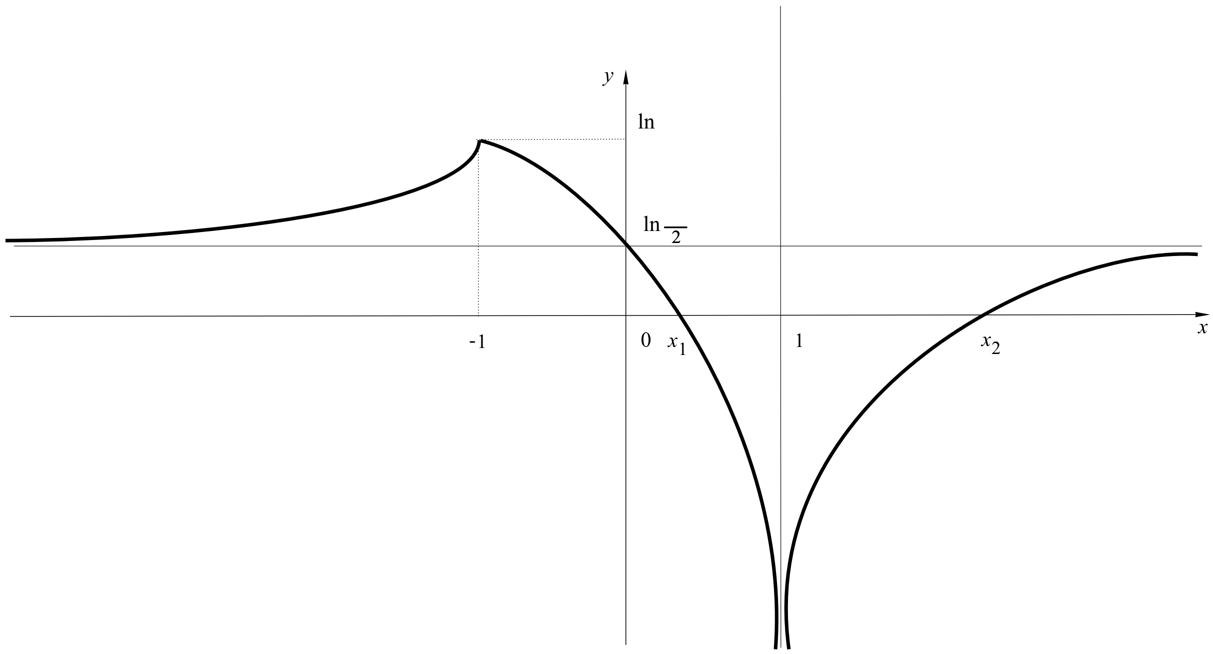
2° $x \in (-1, 0] \Rightarrow x\arccos z - 1 < 0 \Rightarrow y'' < 0$,

3° $x \in (0, 1)$

a) $x \in (0, x_1] \Rightarrow z \in (0, \cos 1] \Rightarrow \arccos z \in [1, \frac{\pi}{2}) \Rightarrow x\arccos z \in (0, \frac{\pi}{2}x_1)$
$\Rightarrow x\arccos z - 1 < \frac{\pi}{2}x_1 - 1 < 0 \Rightarrow y'' < 0$,

b) $x \in (x_1, 1) \Rightarrow z \in (\cos 1, 1) \Rightarrow \arccos z \in (0, 1) \Rightarrow x\arccos z - 1 < 0$
$\Rightarrow y'' < 0$,

4° $x \in (1, \infty) \Rightarrow x\arccos z + 1 > 0 \Rightarrow y'' < 0$.



3. I način. Primitivnu funkciju funkcije $f(x)$ tražimo nad intervalom $I \subset R$ tako da za $x \in I$ važi $x^2 - 3x + 2 = (x-2)(x-1) \neq 0$ i $x^2 - 4x + 3 = (x-3)(x-1) > 0$, odnosno $I \subset (-\infty, 1) \cup (3, \infty)$. Neka je $I \subset (3, \infty)$. Tada Ojlerovom smenom $\sqrt{x^2 - 4x + 3} = \sqrt{(x-3)(x-1)} = t(x-1)$, $x - 3 = t^2(x-1)$, $x = \frac{t^2-3}{t^2-1}$, $dx = \frac{4t}{(t^2-1)^2}dt$, imamo

$$\int f(x)dx = \int \frac{\frac{t^2-3}{t^2-1}}{((\frac{t^2-3}{t^2-1} - 2)(\frac{t^2-3}{t^2-1} - 1)t(\frac{t^2-3}{t^2-1} - 1)} \cdot \frac{4t}{(t^2-1)^2}dt$$

$$= -\int \frac{t^2-3}{t^2+1}dt = -\int (1 - \frac{4}{t^2+1})dt = -t + 4\,\mathrm{arctg}\, t + c$$

$$= -\sqrt{\frac{x-3}{x-1}} + 4\operatorname{arctg}\frac{x-3}{x-1} + c.$$

II način. Uvodeći smenu $x-2 = \frac{1}{cost} > 1$, za $x > 3$, tj. za $t \in (0, \frac{\pi}{2})$, imamo da je $x^2-4x+3 = (x-2)^2-1 = \frac{1}{cos^2 t} - 1 = \frac{\sin^2 t}{cos^2 t}$, $x^2-3x+2 = (x-2)^2+(x-2) = \frac{1+\cos t}{cos^2 t}$, pa je $dx = \frac{\sin t}{cos^2 t}$, i sledi

$$\int f(x)dx = \int \frac{(1+2\cos t)\sin^2 t}{1+\cos t}dt = \int (1+2\cos t)(1-\cos t)dt = \int (1+\cos t - 2\cos^2 t)dt$$

$$= \int (\cos t - \cos 2t)dt = \sin t - \frac{1}{2}\sin 2t + c = \sin(\arccos\frac{1}{x-2}) - \frac{1}{2}\sin(2\arccos\frac{1}{x-2}) + c.$$

III način. Rastavljanjem na parcijalne razlomke $\frac{x}{x^2-3x+2} = \frac{2}{x-2} - \frac{1}{x-1}$, imamo da je integral zadate funkcije za $x > 3$ jednak

$$F(x) = \int f(x)dx = 2\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}} - \int \frac{dx}{(x-1)\sqrt{x^2-4x+3}}.$$

Uvodeći smene $x-2 = \frac{1}{t} > 0$, $dx = -\frac{1}{t^2}dt$ i $x-1 = \frac{1}{s} > 0$, $dx = -\frac{1}{s^2}ds$ sledi

$$F(x) = -2\int \frac{dt}{\sqrt{3t^2-4t+1}} - \int \frac{ds}{\sqrt{3s^2-4s+1}}.$$

Kako je $3t^2-4t+1 = \frac{1}{3}[(3t-2)^2-1] = \frac{1}{3}(m^2-1)$, $m = 3t-2$, $dm = 3dt$, imamo da je $\int \frac{dt}{\sqrt{3t^2-4t+1}} = \frac{\sqrt{3}}{3}\int \frac{dm}{\sqrt{m^2-1}} = \frac{\sqrt{3}}{3}\ln|m+\sqrt{m^2-1}| + c$, te je

$$F(x) = -2\frac{\sqrt{3}}{3}\ln|3t-2+\sqrt{(3t-2)^2-1}| - \frac{\sqrt{3}}{3}\ln|3s-2+\sqrt{(3s-2)^2-1}| + c$$

$$= -\frac{\sqrt{3}}{3}\ln\Big[\Big(\frac{7-2x}{x-2} + \sqrt{(\frac{7-2x}{x-2})^2-1}\Big)^2 |\frac{5-2x}{x-2} + \sqrt{(\frac{5-2x}{x-2})^2-1}|\Big] + c.$$